Learning the Uncertainty Sets for Control Dynamics via Set Membership: A Non-Asymptotic Analysis

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September 28, 2023

Abstract

Set-membership estimation is commonly used in adaptive/learning-based control algorithms that require robustness over the model uncertainty sets, e.g., online robustly stabilizing control and robust adaptive model predictive control. Despite having broad applications, non-asymptotic estimation error bounds in the stochastic setting are limited. This paper provides such a non-asymptotic bound on the diameter of the uncertainty sets generated by set membership estimation on linear dynamical systems under bounded, i.i.d. disturbances. Further, this result is applied to robust adaptive model predictive control with uncertainty sets updated by set membership. We numerically demonstrate the performance of the robust adaptive controller, which rapidly approaches the performance of the offline optimal model predictive controller, in comparison with the control design based on least square estimation's confidence regions.

1 Introduction

The problem of estimating unknown linear dynamical systems of the form $x_{t+1} = A^* x_t + B^* u_t + w_t$ with unknown parameters (A^*, B^*) has seen considerable progress recently, with many works deriving finite-time estimation bound for the unknown system matrices. Most of this literature focuses on the analysis of the least squares estimator (LSE) and its variants. Recently, modern machine learning techniques have been applied to obtain high-probability guarantees of the LSE in finite time Faradonbeh et al. [2018], Sarkar and Rakhlin [2019], Wagenmaker and Jamieson [2020], Mania et al. [2019], Foster et al. [2020], Sarker et al. [2023], where sharp bounds of the convergence rate for LSE were obtained Simchowitz et al. [2018], Foster and Simchowitz [2020], Li et al. [2023].

As a result, there is now a growing literature on "learning to control" unknown linear systems that leverages least-square *point estimation* (LSE), with extensive results for various control objectives such as stability Lale et al. [2022], Kargin et al. [2022] and regret Mania et al. [2019], Dean et al. [2018], Lale et al. [2020]. Such results pioneered modern machine learning techniques for adaptive and data-driven control design. However, in order to apply such learning-based control methods to safety-critical control applications where robust safety requirements such as robust stability and robust constraint satisfaction are commonly required, it is crucial to quantify the uncertainty of the estimated system. The use of an uncertainty set is central to important classes of control methods such as robust model predictive control (RMPC). Too large of an uncertainty set gives rise to conservative control actions, resulting in poor performance. If the uncertainty set is underestimated, the resulting controller may lead to unsafe behavior. Therefore, works have begun to incorporate the confidence region of the LSE for safe adaptive and learning-based control Tanaskovic et al. [2014], Bujarbaruah et al. [2018], Li et al. [2021, 2023].

Parallel to least-square point estimation-based works, another line of research turns to more direct *uncertainty set estimation* methods in the bounded disturbance setting Milanese and Vicino [1991]. Among the algorithms for direct uncertainty set estimation, the set membership (SM) method, which identifies the set of systems models that are consistent with observed data, is one of the most commonly used. The SM method can generate much smaller uncertainty sets than the LSE confidence region, as demonstrated in Figure 1, by leveraging knowledge of the bound on the disturbances. Therefore, numerous robust controller designs and empirical algorithm designs leverage SM for uncertainty set estimation. For example, RMPC algorithms such as Milanese and Vicino [1991], Adetola and Guay
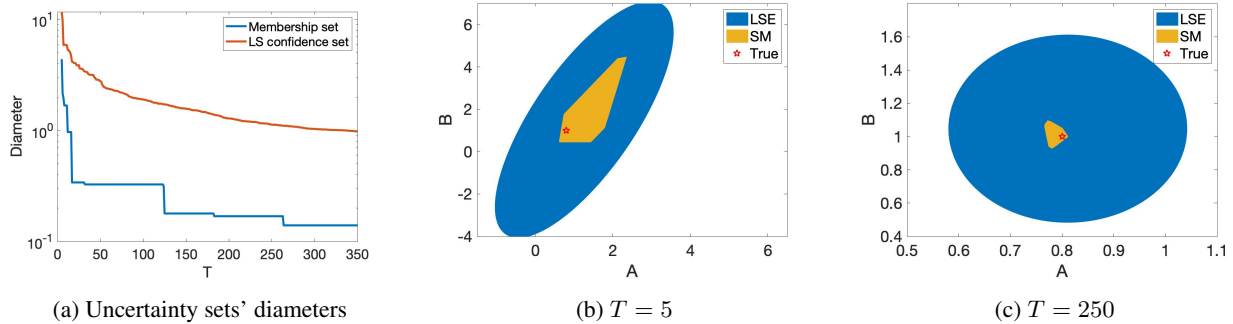
(a) Uncertainty sets' diameters   (b) $T = 5$ (c) $T = 250$

Figure 1: Comparison of the diameters of the membership set of SM and the confidence set generated by LSE for $x_{t+1} = A^* x_t + B^* u_t + w_t$, where $A^* = 0.8$ and $B^* = 1$ are unknown. $w_t \sim \text{TrunGauss}(0, \sigma_w, [-w_{\max}, w_{\max}])$ is i.i.d. and $u_t \sim \text{TrunGauss}(0, \sigma_u, [-u_{\max}, u_{\max}])$ are also i.i.d generated, where $\sigma_w = \sigma_u = 0.5$, $w_{\max} = u_{\max} = 1$. The confidence sets of LSE are computed via Theorem 1 in Abbasi-Yadkori and Szepesvári [2011] with 80% confidence level, where the other parameters (using the notations in Abbasi-Yadkori and Szepesvári [2011]) used in our numerical experiment are: $\lambda = 0.1$, $L = \sigma_w$, $S = \sqrt{(A^*)^2 + (B^*)^2}$.

[2011], Guay et al. [2015], Tanaskovic et al. [2013] have long been employing SM for performance improvement, which is sometimes called robust adaptive MPC and has seen a growing literature recently Lorenzen et al. [2019], Bujarbaruah et al. [2020], Zhang et al. [2021], Parsi et al. [2020a,b], Sasfi et al. [2022]. In addition, SM has also been utilized to stabilize systems despite adversarial disturbances Ho et al. [2021], Yu et al. [2023a], Yeh et al. [2022], Yu et al. [2023b].

On the theory side, there is a long history of theoretical analysis of the SM method for both the deterministic setting where w_t is a deterministic sequence Bai et al. [1995], Fogel and Huang [1982], Kitamura and Fujisaki [2003], Milanese et al. [2013], Lauricella and Fagiano [2020], Eising et al. [2022], Livstone and Dahleh [1996] and the stochastic setting where w_t is independent or even i.i.d. Bai et al. [1998], Lauricella and Fagiano [2020], Akçay [2004], Kitamura et al. [2005], Bai et al. [1995], Lu et al. [2019]. However, in the stochastic setting, most papers consider a simpler regression problem: $y_t = \theta^* x_t + w_t$ with a deterministic sequence of x_t, which does not capture the correlation between x_t and history $w_{t-1}, \ldots, w_0$ in the dynamical systems Bai et al. [1998], Akçay [2004], Kitamura et al. [2005], Bai et al. [1995]. This issue was largely overlooked in the vast empirical algorithm design research related to SM (for example, see Lorenzen et al. [2019], Köhler et al. [2019], etc.). Not until recently, this unsolved correlation problem on SM resurfaced. Work in Lu et al. [2019] provides an initial attempt to establish convergence results on SM for dynamical systems; however, their approach requires a special design of u_t to satisfy persistent excitation requirements deterministically, which can be challenging to satisfy for general control designs in the stochastic settings. Therefore, an open question is:

Can convergence and convergence rates be established for SM on linear dynamical systems under general control design in the stochastic settings?

Contributions. In this paper, to the best of our knowledge, we provide the first finite-time convergence rate bound for the set membership estimation error (Theorem 1) for linear dynamical systems with bounded i.i.d. disturbances. Our bound assumes that the control inputs u_t can guarantee the block-martingale small-ball (BMSB) condition in Simchowitz et al. [2018], which can be satisfied by adding i.i.d. noises to a general class of controllers as discussed in Li et al. [2021, 2023].

Under proper assumptions, we show that the estimation error decays to 0 at a rate of $\tilde{O}(n_x^{1.5}(n_x + n_u)^2 / T)$ when the bound on the disturbances w_t is tight. Interestingly, this is better than the LSE's error bound $O(\frac{\sqrt{n_x + n_u}}{\sqrt{T}})$ in terms of the dependence on the number of samples T but worse in terms of the state and control dimensions n_x, n_u. SM's improvement on the rate with respect to T is enabled by taking advantage of the boundedness of w_t. Even when the bound on w_t is not accurately known but overly estimated, we can show that SM's estimation error converges to a neighborhood of 0 with the same decay rate.

Technically, our estimation error bound relies on a novel construction of an event sequence based on designing a sequence of stopping times. This construction, together with the BMSB condition in Simchowitz et al. [2018], addresses the challenge caused by the correlation between x_t, u_t, and the history disturbances. For more details, we refer the

reader to the proof of Theorem 1.

Moreover, our results lay a foundation for future non-asymptotic analysis of control designs based on SM. To illustrate this, we discuss applications of our results to autonomous system identification and robust-adaptive model predictive control. In each case, we provide numerical comparisons with LSE that demonstrate the promise of SM-based approaches.

2   Problem Formulation and Preliminaries

This paper considers an uncertainty set identification problem for linear dynamical systems with unknown model parameters, i.e.,

$$x_{t+1} = A^* x_t + B^* u_t + w_t \tag{1}$$

where A^*, B^* are unknown, $x_t \in \mathbb{R}^{n_x}, u_t \in \mathbb{R}^{n_u}$. For notational simplicity, we define $\theta^* = (A^*, B^*)$ and $z_t = (x_t^\top, u_t^\top)^\top$, where we write (A, B) as the concatenation of two matrices A, B (the same applies to the vector concatenation). So the system (1) can also be written as $x_{t+1} = \theta^* z_t + w_t$. We define $n_z = n_x + n_u$.

The goal of the uncertainty set identification problem is to determine a set Θ_T that contains the true parameters (A^*, B^*) based on a sequence of data $\{x_t, u_t, x_{t+1}\}_{t=0}^{T-1}$. Set Θ_T is called an uncertainty set since it captures the remaining uncertainty on the system model after the revelation of the data sequence $\{x_t, u_t, x_{t+1}\}_{t=0}^{T-1}$. Uncertainty sets play an important role in robust control, where one aims to achieve robust stability and/or robust constraint satisfaction for any model in the uncertainty set.[1] Therefore, the diameter of the uncertainty sets heavily influences the conservativeness of robust controllers, thus affecting the control performance. Formally, we define the diameter as follows.

Definition 1 (Diameter of a set of matrices). *Consider a set $\mathbb{S}$ of matrices $\theta \in \mathbb{R}^{n_x \times n_z}$. We define the diameter of $\mathbb{S}$ in Frobenius norm as* $\mathrm{diam}(\mathbb{S}) = \sup_{\theta, \theta' \in \mathbb{S}} \|\theta - \theta'\|_F$.

Set Membership.   We focus on a widely adopted uncertainty set identification method: set membership, and provide non-asymptotic bounds for estimation error under proper assumptions.

In particular, we consider bounded disturbances, i.e., there exists a bounded set $\mathcal{W}$ such that $w_t \in \mathcal{W}$ for all $t \geq 0$. When $\mathcal{W}$ is known, the set membership method estimates the uncertainty set by

$$\Theta_T = \bigcap_{t=0}^{T-1} \{\hat{\theta} : x_{t+1} - \hat{\theta} z_t \in \mathcal{W}\}. \tag{2}$$

We also refer to (2) as the *membership set*. Notice that $\theta^* \in \Theta_T$ for all $T \geq 0$ for arbitrary sequence of $w_t \in \mathcal{W}$ without assuming any (stochastic) properties on w_t.

Ideally, one hopes Θ_T converges to the singleton of the true model $\{\theta^*\}$ or at least a small neighborhood of θ^*. This usually calls for additional assumptions, such as the persistent excitation property on the observed data and additional stochastic properties on w_t. In this paper, we consider the following assumptions to establish convergence rate bounds on the diameter of Θ_T.

Assumption 1 (Bounded i.i.d disturbances). *The disturbances are box-constrained, $w_t \in \mathcal{W} := \{w \in \mathbb{R}^{n_x} : \|w\|_\infty \leq w_{\max}\}$ for all $t \geq 0$. Further, w_t is i.i.d.. Let μ_w and $\Sigma_w \succ 0$ denote the mean and covariance of w_t.*

In many applications, it is realistic to assume the disturbances are bounded, e.g., wind perturbation for aerial robotics and temperature fluctuations for building control. Therefore, it is advantageous to leverage such prior knowledge about the disturbances for system estimation and control design beyond the sub-Gaussian stochasticity assumption commonly seen in the system identification literature.

Next, we introduce the assumptions on u_t, which relies on the block-martingale small-ball condition proposed in Simchowitz et al. [2018]. It can be shown that the block-martingale small-ball condition can guarantee persistent excitation with high probability under proper conditions (see [Simchowitz et al., 2018, Proposition 2.5] and Lemma 1). In the system identification literature, the persistent excitation (PE) condition is essential for ensuring that the estimation error decays. The PE condition essentially ensures that $(x_t^\top, u_t^\top)$ explores all directions of the state space.

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[1] In addition to model uncertainties, robust control may also consider other system uncertainties, e.g., disturbances, measurement noises, etc.

Definition 2 (Persistent excitation). *There exists $\alpha > 0$ and $m \in \mathbb{N}+$, such that for any $t_0 \geq 0$,*

$$\frac{1}{m} \sum_{t=t_0}^{t_0+m-1} \begin{pmatrix} x_t \\ u_t \end{pmatrix} (x_t^\top, u_t^\top) \succeq \alpha^2 I_{n_x+n_u}.$$

Definition 3 (BMSB Simchowitz et al. [2018]). *Let $\{\mathcal{F}_t\}_{t \geq 1}$ denote a filtration and let $\{Z_t\}_{t \geq 1}$ be an $\{\mathcal{F}_t\}_{t \geq 1}$-adapted random process taking values in $\mathbb{R}^d$. We say that $\{Z_t\}_{t \geq 1}$ satisfies the (k, Γ_{sb}, p)-block martingale small-ball (BMSB) condition for a positive integer k, a positive definite matrix $\Gamma_{sb} \succ 0$, and $0 \leq p \leq 1$, if the following condition holds: for any fixed $\lambda \in \mathbb{R}^d$ such that $\|\lambda\|_2 = 1$, the process $\{Z_t\}_{t \geq 1}$ satisfies $\frac{1}{k} \sum_{i=1}^{k} \mathbb{P}(|\lambda^\top Z_{t+i}| \geq \sqrt{\lambda^\top \Gamma_{sb} \lambda} \mid \mathcal{F}_t) \geq p$ almost surely for any $t \geq 1$.*

Our bound depends on the following assumption.

Assumption 2 (BMSB and boundedness). *There exists $b_z \geq 0$ such that $\|z_t\|_2 \leq b_z$ almost surely for all $t \geq 0$. Further, considering filtration $\mathcal{F}_t = \mathcal{F}(w_0, \ldots, w_{t-1}, z_0, \ldots, z_t)$, then the $\mathcal{F}_t$-adapted stochastic process $\{z_t\}_{t \geq 0}$ satisfies $(1, \sigma_z^2 I_{n_z}, p_z)$-BMSB for some $p_z > 0$.*

Assumption 2 can be satisfied in many scenarios. For example, the boundedness of z_t can be achieved by robust stabilizing controllers Tu [2019], Yu et al. [2023a] and robust constraint-satisfying controllers with bounded constraint sets, such as robust model predictive controllers, e.g., Rawlings et al. [2017]. Further, the BMSB condition can be achieved by inserting a bounded i.i.d. random noise to any control policies π_t, i.e., $u_t = \pi_t(x_t) + \eta_t$, where η_t and w_t have positive definite covariance matrices [Li et al., 2021, Theorem 1].

Finally, we assume that the bound $w_{\max}$ on w_t is tight in all directions.

Assumption 3 (Tight bound on w_t). *For any $\epsilon > 0$, there exists $q_w(\epsilon) > 0$, such that for any $1 \leq j \leq n$ and $t \geq 0$, we have*

$$\mathbb{P}(w_t^j + w_{\max} \leq \epsilon) \geq q_w(\epsilon) > 0, \quad \mathbb{P}(w_{\max} - w_t^j \leq \epsilon) \geq q_w(\epsilon) > 0.$$

Assumption 3 holds on many different types of distributions, e.g., uniform distributions on $\mathcal{W}$ or truncated Gaussian distributions on $\mathcal{W}$. For example, it can be verified that for uniform distributions on $\mathcal{W}$, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}}$; and for truncated Gaussian distributions with 0 mean, covariance $\sigma_w^2 I_n$, and truncated region $\mathcal{W}$, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}\sigma_w} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})$ (see appendix for formal proofs).[2] Another common scenario is that even though w_t is bounded by $\mathcal{W}$, the system designer may only know a loose upper bound $\hat{w}_{\max} \geq w_{\max}$. In this case, we will show that the membership set converges to a small region around θ^* with the same convergence rate.

Remark 1 (Comparison with least square estimation). *On the one hand, compared with the assumptions for least square estimation's convergence rate, set membership requires additional assumptions on bounded $\mathcal{W}$ and tight bound $w_{\max}$ to establish convergence rate. On the other hand, if we do not aim for convergence (rates), SM generates valid uncertainty sets, i.e., $\theta^* \in \Theta_T$, even without any stochastic assumptions, while LSE's uncertainty set is determined based on confidence region, which only holds under proper stochastic properties.*

3 Set Membership Convergence Analysis

We now present the main result (Theorem 1) of this paper, which is a non-asymptotic bound on the estimation errors of the SM method given bounded i.i.d. stochastic disturbances.

Theorem 1 (Diameter bound on the membership set). *For any $m > 0$ any $\delta > 0$, when $T > m$, we have*

$$\mathbb{P}(\text{diam}(\Theta_T) > \delta) \leq \underbrace{\frac{T}{m} \tilde{O}(n_z^{2.5}) a_2^{n_z} \exp(-a_3 m)}_{\text{Term 1}} + \underbrace{\tilde{O}((n_x n_z)^{2.5}) a_4^{n_x n_z} \left(1 - q_w \left(\frac{a_1 \delta}{4\sqrt{n_x}}\right)\right)^{\lceil T/m \rceil}}_{\text{Term 2}}$$

where $a_1 = \frac{\sigma_z p_z}{4}$, $a_2 = \frac{64 b_z^2}{\sigma_z^2 p_z^2}$, $a_3 = \frac{p_z^2}{8}$, $a_4 = \max(\frac{4 b_z \sqrt{n_x}}{a_1}, 1)$, $p_z, \sigma_z, w_{\max}, b_z$ are as defined in Assumption 1-2, and $\lceil \cdot \rceil$ denotes the ceiling function.

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[2]Throughout the paper, we use $\text{TrunGauss}(0, \sigma_w, [-w_{\max}, w_{\max}])$ to refer to the truncated Gaussian distribution generated by Gaussian distribution with $\text{Gauss}(0, \sigma_w^2)$ with truncated range $[-w_{\max}, w_{\max}]$.

Theorem 1 provides an upper bound on the "failure" probability of the SM method, i.e., the probability that the diameter of the membership set is larger than δ. In this bound, Term 1 decays exponentially with m, so for any small $\epsilon > 0$, m can be chosen such that Term 1 $\leq \epsilon$, which indicates $m \geq O(n_z + \log T + \log(1/\epsilon))$. Term 2 decays exponentially with the number of data points T and involves a distribution-dependent function $q_w(\cdot)$, which characterizes how likely it is for w_t to visit the boundary of $\mathcal{W}$ as defined in Assumption 3. If w_t is more likely to visit the boundary, i.e. $q_w(\cdot)$ is larger, then SM method is less likely to generate an uncertainty set with a diameter bigger than δ.

Estimation error bounds when $q_w(\epsilon) = O(\epsilon)$. To provide intuition for Term 2 and discuss the estimation error bound in Theorem 1 more explicitly, we consider distributions satisfying $q_w(\epsilon) = O(\epsilon)$ for all $\epsilon > 0$. Notice that several common distributions satisfy this additional requirement, such as uniform distribution and truncated Gaussian distribution as discussed after Assumption 3.

Corollary 1 (Estimation error bound when $q_w(\epsilon) = O(\epsilon)$). *For any $\epsilon > 0$, let*

$$m \geq O(n_z + \log T + \log(1/\epsilon)).$$

If w_t is generated i.i.d. by a distribution satisfying $q_w(\epsilon) = O(\epsilon)$ for all $\epsilon > 0$, then with probability at least $1 - 2\epsilon$, for any $\hat{\theta}_T \in \Theta_T$, we have

$$\|\hat{\theta}_T - \theta^*\|_F \leq \text{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}(n_x + n_u)^2}{T}\right).$$

Corollary 1 indicates that the estimation error of any point in the membership set Θ_T can be bounded by $\tilde{O}\left(\frac{n_x^{1.5}(n_x+n_u)^2}{T}\right)$ when $q_w(\epsilon) \geq O(\epsilon)$.

Loose Bound on $w_{\max}$. In many practical scenarios, it is easier to obtain an over-estimation of the range of the disturbances instead of a tight upper bound. Hence, we discuss the scenarios when we only know an over-estimation $\hat{w}_{\max} \geq w_{\max}$. In this case, we can further show that the membership set converges to a small neighborhood around θ^* with the same convergence rate as Theorem 1.

Theorem 2 (Loose bound on $w_{\max}$). *Consider the membership set*

$$\hat{\Theta}_T = \bigcap_{t=0}^{T-1}\{\hat{\theta} : \|x_{t+1} - \hat{\theta}z_t\|_\infty \leq \hat{w}_{\max}\},$$

where $\hat{w}_{\max}$ is a loose upper bound for the disturbances such that $\hat{w}_{\max} \geq w_{\max}$. There exists a constant $\epsilon_0 = O((\hat{w}_{\max} - w_{\max})\sqrt{n_x})$ such that for any $m > 0$, $\delta > 0$, $T > m$, we have

$$\mathbb{P}(\text{diam}(\hat{\Theta}_T) > \delta + \epsilon_0) \leq \text{Term 1} + \text{Term 2}$$

where Term1 *and* Term2 *are defined in Theorem 1.*

Similarly to Corollary 1, we can establish $\|\hat{\theta}_T - \theta^*\|_F \leq \epsilon_0 + \tilde{O}(n_x^{1.5}n_z^2/T)$ when $q_w(\epsilon) = O(\epsilon)$ for any $\epsilon > 0$. This establishes a convergence rate towards an ϵ_0-neighborhood around θ^*.

3.1   Proof Sketch of Theorem 1

A formal proof of Theorem 1 is provided in the supplementary material. Here, we provide a sketch to highlight the key technical ideas of the proof.

Specifically, we first define a set Γ_T on the model estimation error $\gamma = \hat{\theta} - \theta^*$ by leveraging the observation that $x_{s+1} - \hat{\theta}z_s = w_s - (\hat{\theta} - \theta^*)z_s$,

$$\Gamma_t = \bigcap_{s=0}^{t-1}\{\gamma : \|w_s - \gamma z_s\|_\infty \leq w_{\max}\}, \quad \forall\, t \geq 0. \tag{3}$$

Notice that $\Theta_t = \theta_* + \Gamma_t$, so $\mathrm{diam}(\Theta_t) = \mathrm{diam}(\Gamma_t)$, and

$$\mathrm{diam}(\Gamma_t) = \sup_{\gamma, \gamma' \in \Gamma_t} \|\gamma - \gamma'\|_F \le 2 \sup_{\gamma \in \Gamma_t} \|\gamma\|_F.$$

Therefore, we can upper bound our goal event $\{\mathrm{diam}(\Theta_T) > \delta\}$ by the event $\mathcal{E}_1$ defined below.

$$\mathbb{P}(\mathrm{diam}(\Theta_T) > \delta) \le \mathbb{P}(\mathcal{E}_1), \text{ where } \mathcal{E}_1 := \{\exists\, \gamma \in \Gamma_T, \text{ s.t. } \|\gamma\|_F \ge \frac{\delta}{2}\}. \tag{4}$$

Next, we define an event $\mathcal{E}_2$ below, which is essentially persistent excitation (PE) on every time segments $km \le t \le km + m - 1$ for $k \ge 0$, where the choice of m will be specified later.

$$\mathcal{E}_2 = \{\frac{1}{m} \sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}, \ \forall\, 0 \le k \le \lceil T/m \rceil - 1\}$$

where $a_1 = \frac{\sigma_z p_z}{4}$. For ease of notation and without loss of generality, we assume T/m is an integer in the following analysis. The proof can now be completed by taking the intersection of $\mathcal{E}_1$ with $\mathcal{E}_2$ and $\mathcal{E}_2^c$ in (4). Specifically, from (4) we have

$$\mathbb{P}(\mathrm{diam}(\Theta_T) > \delta) \le \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2^c) + \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \le \mathbb{P}(\mathcal{E}_2^c) + \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2).$$

The proof concludes by invoking the following bounds on $\mathbb{P}(\mathcal{E}_2^c)$ and $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2)$.

Lemma 1 (Bound on $\mathbb{P}(\mathcal{E}_2^c)$)**.**

$$\mathbb{P}(\mathcal{E}_2^c) \le \frac{T}{m} \tilde{O}(n_z^{2.5}) a_2^{n_z} \exp(-a_3 m)$$

where $a_2 = \frac{64 b_z^2}{\sigma_z^2 p_z^2}$ and $a_3 = \frac{p_z^2}{8}$.

Lemma 2 (Bound on $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2)$)**.**

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \le \tilde{O}(n_z^{2.5} n_x^{2.5}) a_4^{n_x n_z} (1 - q_w(\frac{a_1 \delta}{4\sqrt{n_x}}))^{T/m}$$

where $a_4 = \max(1, 4 b_z \sqrt{n_x}/a_1)$ and $n_z = n_x + n_u$.

Roughly speaking, Lemma 1 indicates that PE holds with high probability. This is proved by leveraging the BMSB assumption and set discretization. The proof of Lemma 2 is more involved and is our major technical contribution. On a high level, the proof relies on the following two technical lemmas.

Lemma 3 (Discretization of $\mathcal{E}_1 \cap \mathcal{E}_2$ (Informal))**.** *Let $\mathcal{M} = \{\gamma_1, \dots, \gamma_{v_\gamma}\}$ denotes an ϵ_γ-net of $\{\gamma : \|\gamma\|_F = 1\}$. Under a proper choice of ϵ_γ, we have $v_\gamma = \tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}$, and we can construct $\tilde{\Gamma}_T$,[3] such that*

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \le \mathbb{P}(\{\exists\, 1 \le i \le v_\gamma, d \ge 0, \text{ s.t. } d\gamma_i \in \tilde{\Gamma}_T\} \cap \mathcal{E}_2) \le \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2)$$

where $\mathcal{E}_{1,i} = \{\exists\, d \ge 0, \text{ s.t. } d\gamma_i \in \tilde{\Gamma}_T\}$.

Lemma 3 leverages finite set discretization to bound the existence of a feasible element in an infinite continuous set.

Lemma 4 (Construction of event $G_{i,k}$ via stopping times)**.** *Under the conditions in Lemma 3, we can construct $G_{i,k}$ for all i and all $0 \le k \le T/m - 1$ by*

$$G_{i,k} = \{b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} \ge \frac{a_1 \delta}{4\sqrt{n_x}} - w_{\max}, \text{ and } \frac{1}{m} \sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}\}.$$

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[3]In the Appendix, the accurate version of this lemma will specify the choice of $\tilde{\Gamma}_T$.

where $1 \leq L_{i,k} \leq m$ is constructed as a stopping time with respect to $\{\mathcal{F}_{km+l}\}_{l \geq 0}$ and $b_{i,t}, j_{i,t} \in \mathcal{F}_t$.
It can be shown that

$$\mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{T/m-1} G_{i,k}) = \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0})\cdots\mathbb{P}(G_{i,T/m-1} \mid \bigcap_{k=0}^{T/m-2} G_{i,k})$$

and

$$\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}).$$

The construction of $G_{i,k}$ in Lemma 4 is our major technical contribution. Notice that by constructing a stopping time $L_{i,k}$, conditioning on $L_{i,k} = l$, $w_{km+L_{i,k}} = w_{km+l}$ is independent of $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$, and $b_{i,km+L_{i,k}}, j_{i,km+L_{i,k}}$ are constant conditioning on $\mathcal{F}_{km+l}$. Hence, we are able to leverage the properties of the probability distribution of w_t in Assumption 3 to bound $\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$.

By leveraging Lemma 3 and 4, we can now prove Lemma 2. Then, the proof of Theorem 1 is completed by combining Lemma 1 and Lemma 2.

4 Applications

To showcase the results in the previous sections, we apply them in the context of two applications. The first application is the estimation problem of a linear control dynamical system with random control inputs. The second is the analysis of an adaptive tube model predictive control (MPC) algorithm.

4.1 Estimating Dynamical Systems

We now apply Theorem 1 to linear systems with no control inputs: $x_{t+1} = A^*x_t + w_t$. In this setting, the membership set can be defined as

$$\mathbb{A}_T = \bigcap_{t=0}^{T-1}\{\hat{A} : \|x_{t+1} - \hat{A}x_t\|_\infty \leq w_{\max}\}.$$

We will provide a diameter upper bound on $\mathbb{A}_T$ below.

Corollary 2 (System estimation when $B = 0$). *When A^* is (κ, ρ)-stable, i.e., $\|(A^*)^t\|_2 \leq \kappa(1-\rho)^t$ for all t with $\rho < 1$, for any $m > 0$ and any $\delta > 0$, when $T > m$, we have*

$$\mathbb{P}(\text{diam}(\mathbb{A}_T) > \delta) \leq \frac{T}{m}\tilde{O}(n_x^{2.5})a_2^{n_x}\exp(-a_3m) + \tilde{O}(n_x^5)a_4^{n_x^2}(1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{\lceil T/m \rceil}$$

where $b_x = \kappa\|x_0\|_2 + \kappa\sqrt{n_x}/\rho$, $p_x = 1/192$, $\sigma_x = \sqrt{\lambda_{\min}(\Sigma_w)/2}$, $a_1 = \frac{\sigma_x p_x}{4}$, $a_2 = \frac{64w_{\max}}{\sigma_x^2 p_x^2}$, $a_3 = \frac{p_x^2}{8}$, $a_4 = \max(\frac{4b_x\sqrt{n_x}}{a_1}, 1)$.

Consequently, when the distribution of w_t satisfies $q_w(\epsilon) = O(\epsilon)$, e.g. uniform or truncated Gaussian, we have $\|\hat{\theta} - \theta_*\| \leq \tilde{O}(n_x^{3.5}/T)$.

Corollary 2 can be proved by verifying the assumptions of Theorem 1: we can show $\|x_t\|_2 \leq b_x$ for stable A^* and prove the $(1, \sigma_x, p_x)$-BMSB condition for $\{x_t\}_{t \geq 0}$ based on Dean et al. [2019a].

Interestingly, though Simchowitz et al. [2018] shows that $\Omega(\sqrt{n_x}/\sqrt{T})$ is the best possible estimation rate for (unbounded) Gaussian distribution, when the random w_t are *bounded*, Corollary 2 shows that SM can achieve a better rate $\tilde{O}(n_x^3/T)$ when $T >> n_x$ by taking advantage of the additional information of the bounded support. The fundamental lower bound for bounded random w_t are left for future work.

It is worth comparing this upper bound with the estimation error bound of LSE in Dean et al. [2019a], which is $\tilde{O}(\frac{\sqrt{n_x}}{\sqrt{T}})$. Interestingly, the point estimators generated by the SM method achieve better convergence rates in terms of the number of samples T, but are worse with respect to the system dimensions n_x, n_u.

4.2 Adaptive Tube Model Predictive Control

The set membership method is extensively used in the robust adaptive model predictive control (RA-MPC) literature, e.g., Tanaskovic et al. [2014], Lu and Cannon [2023]. In RA-MPC, the model uncertainty sets can be updated online based on set membership estimation as more data is collected. Since the model uncertainty sets heavily influence the conservativeness of RA-MPC controllers, SM updates are able to quickly improve control performance by reducing the size of the uncertainty/membership sets.

In the following, we provide a brief recap of RA-MPC and show the implication of our SM result. At every time step k, RA-MPC solves the following robust optimization with horizon H:

$$
\begin{aligned}
&\min_{\{\pi_{k+t|k}\}_{t=0}^{H}} \sum_{t=0}^{H} c_t(\hat{x}_{k+t|k}, \hat{u}_{k+t|k}) \\
\text{s.t.} \quad & \hat{x}_{k+t+1|k} = \hat{A}_k \hat{x}_{k+t|k} + \hat{B}_k \hat{u}_{k+t|k}, \quad t \geq 0, \\
& \hat{u}_{k+t|k} = \pi_{k+t|k}\left(\hat{x}_{k+t|k}\right), \quad t \geq 0, \\
& \tilde{x}_{k+t+1|k} = \tilde{A}\tilde{x}_{k+t|k} + \tilde{B}\tilde{u}_{k+t|k} + \tilde{w}_{k+t|k}, \; \forall (\tilde{A}, \tilde{B}) \in \Theta_k, \tilde{w}_{k+t|k} \in \mathcal{W}, t \geq 0 \\
& \tilde{u}_{k+t|k} = \pi_{k+t|k}\left(\tilde{x}_{k+t|k}\right), \quad t \geq 0, \\
& \hat{x}_{k|k} = \tilde{x}_{k|k} = x_k, \\
& \tilde{x}_{k+t|t} \in \mathcal{X}, \; \tilde{u}_{k+t|t} \in \mathcal{U}, \; x_{k+H|t} \in \mathcal{X}_H, \; t = 0, 1, \cdots, H-1
\end{aligned}
\tag{5}
$$

with respect to some cost objective c_t. It is evident in (5) that smaller uncertainty set Θ_t leads to better performance of the MPC algorithm. In particular, we adaptively select the center of the uncertainty set Θ_k as the nominal model $\hat{A}_k$, $\hat{B}_k$ in (5). For more details of RAMPC, one can refer to the literature Tanaskovic et al. [2014], Bujarbaruah et al. [2020], Lu et al. [2019], Lu and Cannon [2023], etc.

There is a vast literature of algorithms that approximate the solutions to (5) for fixed uncertainty sets, e.g., Mayne et al. [2005], Rakovic et al. [2016], Chen et al. [2021]. We will leverage such methods to solve (5) for the adaptive control task. For concreteness, we consider the basic adaptive tube MPC algorithm Rawlings et al. [2017], Mayne et al. [2005] for the FTCOP problem here. We parameterize the control policy π as $u_k = Kx_k + v_k + \eta_k$, where K is a given robustly stabilizing feedback gain for the entire uncertainty set $\mathcal{P}_0$ (and thus for all subsequent $\mathcal{P}_k$), v_k is the FTCOP optimization variable, and η_k is a bounded exploration injection with $\|\eta_k\|_\infty \leq \eta_{\max}$ with $\mathbb{E}[\eta_k \eta_k^\top] = \sigma_\eta I$. Note that η_k is left out for (5).

Applying Theorem 1, we obtain a finite-time convergence guarantee for this RA-MPC algorithm that integrates SM with tube MPC.

Corollary 3. *Assume that* (5) *is feasible for $\mathcal{P}_0$ and Assumption 3 holds, the membership set under the adaptive tube MPC controller has the following uncertainty set convergence guarantee, $\|\hat{\theta}_T - \theta^*\|_F \leq \tilde{O}(\frac{n_x^{3.5} n_u^2}{T})$ with high probability.*

This result is immediate by verifying Assumption 2 using Li et al. [2023] and verifying the boundedness assumption by noting that $\mathcal{X}, \mathcal{U}$ are bounded.

5 Numerical Experiments

Uncertainty sets comparison for control dynamic systems with i.i.d. inputs. To illustrate the quantitative behavior of Corollary 1, we provide simulation results in Figure 2(a)-(b). Our experiments illustrate the membership set diameter as a function of time for a control dynamical system $x_{t+1} = A^* x_t + B^* u_t + w_t$, where $A \in \mathbb{R}^{10 \times 10}$, $B \in \mathbb{R}^{10 \times 3}$, and w_t is drawn i.i.d from (a) projected Gaussian and (b) uniform distribution with $w_{\max} = 2$. We apply SM with differently conservative upper bounds on $w_{\max}$, i.e., $\hat{w}_{\max} = w_{\max}, 2w_{\max}, 5w_{\max}, 10w_{\max}$. We compute LSE's confidence region by Theorem 1 in Abbasi-Yadkori et al. [2011] with parameters $\delta = 0.1$ (90% confidence), $\lambda = 0.1$ (LSE regularization parameter), $S = \sqrt{\text{trace}((\theta^*)^\top \theta^*)}$, and $L = 1$ for projected Gaussian and $L = 4/3$ for Uniform distribution.

The quantitative behavior of the SM method is consistent with that predicted by Corollary 1, outperforming the 90% confidence region of the LSE in both magnitude and rate of convergence when the disturbance bound is tight. Moreover, even when the bound on w_t is loose, i.e., Assumption 3 fails to hold, SM is still orders of magnitude better than the LSE in terms of the size of the uncertainty set.

(a) Projected Gaussian

(b) Uniform Distribution
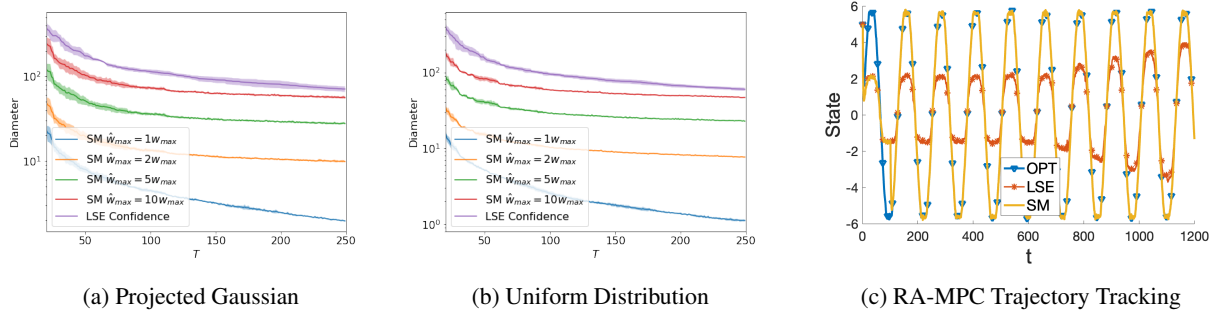
(c) RA-MPC Trajectory Tracking

Figure 2: **Figure (a)-(b)** compares the diameters of SM and LSE's uncertainty sets with loose disturbance bounds for SM. **Figure (c)** demonstrates the tracking performance of an adaptive robust model predictive controller based on LSE confidence sets and membership sets.

Tracking performance comparison by RAMPC with different set estimation methods. To illustrate the quantitative impact of using SM for adaptive tube MPC, we provide simulation results in Figure 2. Our experiments study tube MPC for a single-input-single-output system with nominal system $A^* = 1.2$, $B^* = 0.9$ with an initial model uncertainty set $\Theta_0 := [1, 1.2] \times [0.9, 1.1]$. We use the basic tube MPC method Rawlings et al. [2017], Mayne et al. [2005] and parameterize the control policy as $u_k = Kx_k + v_k + \eta_k$, where $K = -1$ and η_k is a bounded exploration injection with $\eta_k \sim \text{Unif}([-0.01, 0.01])$. Note that η_k is left out when solving for (5). The disturbance w_k has a known bound of $w_{\max} = 0.1$ and is generated to be i.i.d. $\text{Unif}([-0.1, 0.1])$. The horizon of (5) is set to be 5. The state and input constraints are such that $x_k \in [-10, 10]$ and $u_k \in [-10, 10]$ for all $k \geq 0$. We consider the task of constrained LQ tracking problem with a time-varying cost function $c_t := (x_t - g_t)^\top Q(x_t - g_t) + u_t^\top R u_t$ where the target trajectory is generated as $g_t = 8\sin(t/20)$.

We compare the performance of an adaptive tube MPC controller that uses the SM method for uncertainty set estimation (SM) against one that uses the LSE confidence region (LSE). We also plot the offline optimal RMPC controller, i.e., the controller that has knowledge of the true underlying system parameters (OPT). The resulting state trajectories are shown in Figure 2c. Due to the fast convergence of the membership set, the tracking performance of SM quickly coincides with OPT, while LSE converges more slowly. Since the controller has to robustly satisfy constraints against the worst-case model in the uncertainty set, smaller uncertainty set for (5) means more optimal trajectories are available to the RA-MPC controller. Therefore, Corollary 3 verifies that the SM method is indeed a good candidate for the uncertainty set estimation problem arising in RA-MPC problems as have been extensively observed empirically in existing literature Lorenzen et al. [2019], Lu et al. [2019], Köhler et al. [2019].

6   Conclusion, Limitations, and Future Work

Conclusion.   This work provides an explicit bound on the convergence rate of set membership estimation methods for linear control dynamical systems under the assumption of i.i.d. stochastic disturbances that are bounded by box constraints. When the box constraints are accurately known, we establish the rate of convergence to the true model parameters. When the box constraints on w_t are over-estimated, we establish the rate of convergence to a neighborhood of the true model parameters. We demonstrate the main result of this paper (Theorem 1) in both the linear system identification problem and the adaptive robust model predictive control setting. Numerical experiments on these two settings are also provided in comparison with least square estimation and its confidence region.

Limitations and future work.   One limitation of our work is that our bound heavily relies on the assumption of the box constraints on the disturbances, which may be overly conservative in practice. Therefore, an important future direction is to analyze set membership estimation for general polyhedron constraints on the disturbances. Moreover, our results have focused exclusively on the *diameter* of the membership set. Other metrics such as the volume are also of importance in the control literature and therefore needs more investigation. For example, if the uncertainty set has a shape similar to a narrow ellipse, it may have a large diameter yet small volume, and whether this has high impact or low impact on control performance is also worth discussing. Another limitation of this work is that we only consider

linear systems, and it would be interesting to explore whether some of the results in this paper can be extended to some nonlinear systems, such as bi-linear systems, or nonlinear systems with linear parametrization, etc.

Interesting directions for future work also include obtaining a lower bound for set membership estimation. Further, we note that Corollary 3 also lays a foundation for non-asymptotic optimality analysis for RA-MPC based on set membership, which has seen a significant amount of empirical research but lacks formal optimality analysis. Finally, it is also worth applying the proof techniques of this paper to analyze the set membership performance on other types of uncertainties, such as measurement errors, etc.

References

Mohamad Kazem Shirani Faradonbeh, Ambuj Tewari, and George Michailidis. Finite-time adaptive stabilization of linear systems. *IEEE Transactions on Automatic Control*, 64(8):3498–3505, 2018.

Tuhin Sarkar and Alexander Rakhlin. Near optimal finite time identification of arbitrary linear dynamical systems. In *International Conference on Machine Learning*, pages 5610–5618. PMLR, 2019.

Andrew Wagenmaker and Kevin Jamieson. Active learning for identification of linear dynamical systems. In *Conference on Learning Theory*, pages 3487–3582. PMLR, 2020.

Horia Mania, Stephen Tu, and Benjamin Recht. Certainty equivalence is efficient for linear quadratic control. In *Advances in Neural Information Processing Systems*, volume 32, pages 10154–10164. Curran Associates, Inc., 2019.

Dylan Foster, Tuhin Sarkar, and Alexander Rakhlin. Learning nonlinear dynamical systems from a single trajectory. In *Learning for Dynamics and Control*, pages 851–861. PMLR, 2020.

Arnab Sarker, Peter Fisher, Joseph E Gaudio, and Anuradha M Annaswamy. Accurate parameter estimation for safety-critical systems with unmodeled dynamics. *Artificial Intelligence*, page 103857, 2023.

Max Simchowitz, Horia Mania, Stephen Tu, Michael I Jordan, and Benjamin Recht. Learning without mixing: Towards a sharp analysis of linear system identification. In *Conference On Learning Theory*, pages 439–473. PMLR, 2018.

Dylan J Foster and Max Simchowitz. Logarithmic regret for adversarial online control. *arXiv preprint arXiv:2003.00189*, 2020.

Yingying Li, Tianpeng Zhang, Subhro Das, Jeff Shamma, and Na Li. Non-asymptotic system identification for linear systems with nonlinear policies. *arXiv preprint arXiv:2306.10369*, 2023.

Sahin Lale, Kamyar Azizzadenesheli, Babak Hassibi, and Animashree Anandkumar. Reinforcement learning with fast stabilization in linear dynamical systems. In *International Conference on Artificial Intelligence and Statistics*, pages 5354–5390. PMLR, 2022.

Taylan Kargin, Sahin Lale, Kamyar Azizzadenesheli, Animashree Anandkumar, and Babak Hassibi. Thompson sampling achieves $\tilde{o}\sqrt{t}$ regret in linear quadratic control. In *Conference on Learning Theory*, pages 3235–3284. PMLR, 2022.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In *Advances in Neural Information Processing Systems*, pages 4188–4197, 2018.

Sahin Lale, Kamyar Azizzadenesheli, Babak Hassibi, and Anima Anandkumar. Explore more and improve regret in linear quadratic regulators. *arXiv preprint arXiv:2007.12291*, 2020.

Marko Tanaskovic, Lorenzo Fagiano, Roy Smith, and Manfred Morari. Adaptive receding horizon control for constrained mimo systems. *Automatica*, 50(12):3019–3029, 2014.

Monimoy Bujarbaruah, Xiaojing Zhang, and Francesco Borrelli. Adaptive mpc with chance constraints for fir systems. In *2018 Annual American Control Conference (ACC)*, pages 2312–2317. IEEE, 2018.

Yingying Li, Subhro Das, Jeff Shamma, and Na Li. Safe adaptive learning-based control for constrained linear quadratic regulators with regret guarantees. *arXiv preprint arXiv:2111.00411*, 2021.

Mario Milanese and Antonio Vicino. Optimal estimation theory for dynamic systems with set membership uncertainty: An overview. *Automatica*, 27(6):997–1009, 1991.

Veronica Adetola and Martin Guay. Robust adaptive mpc for constrained uncertain nonlinear systems. *International Journal of Adaptive Control and Signal Processing*, 25(2):155–167, 2011.

Martin Guay, Veronica Adetola, and Darryl DeHaan. *Robust and adaptive model predictive control of nonlinear systems*. Institution of Engineering and Technology, 2015.

Marko Tanaskovic, Lorenzo Fagiano, Roy Smith, Paul Goulart, and Manfred Morari. Adaptive model predictive control for constrained linear systems. In *2013 European Control Conference (ECC)*, pages 382–387. IEEE, 2013.

Matthias Lorenzen, Mark Cannon, and Frank Allgöwer. Robust mpc with recursive model update. *Automatica*, 103: 461–471, 2019.

Monimoy Bujarbaruah, Xiaojing Zhang, Marko Tanaskovic, and Francesco Borrelli. Adaptive stochastic mpc under time-varying uncertainty. *IEEE Transactions on Automatic Control*, 66(6):2840–2845, 2020.

Kunwu Zhang, Qi Sun, and Yang Shi. Trajectory tracking control of autonomous ground vehicles using adaptive learning mpc. *IEEE Transactions on Neural Networks and Learning Systems*, 32(12):5554–5564, 2021.

Anilkumar Parsi, Andrea Iannelli, Mingzhou Yin, Mohammad Khosravi, and Roy S Smith. Robust adaptive model predictive control with worst-case cost. *IFAC-PapersOnLine*, 53(2):4222–4227, 2020a.

Anilkumar Parsi, Andrea Iannelli, and Roy S Smith. Active exploration in adaptive model predictive control. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 6186–6191. IEEE, 2020b.

András Sasfi, Melanie N Zeilinger, and Johannes Köhler. Robust adaptive mpc using control contraction metrics. *arXiv preprint arXiv:2209.11713*, 2022.

Dimitar Ho, Hoang Le, John Doyle, and Yisong Yue. Online robust control of nonlinear systems with large uncertainty. In *International Conference on Artificial Intelligence and Statistics*, pages 3475–3483. PMLR, 2021.

Jing Yu, Dimitar Ho, and Adam Wierman. Online adversarial stabilization of unknown networked systems. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, 7(1):1–43, 2023a.

Christopher Yeh, Jing Yu, Yuanyuan Shi, and Adam Wierman. Robust online voltage control with an unknown grid topology. In *Proceedings of the Thirteenth ACM International Conference on Future Energy Systems*, pages 240–250, 2022.

Jing Yu, Varun Gupta, and Adam Wierman. Online stabilization of unknown linear time-varying systems. *arXiv preprint arXiv:2304.02878*, 2023b.

Yasin Abbasi-Yadkori and Csaba Szepesvári. Regret bounds for the adaptive control of linear quadratic systems. In *Proceedings of the 24th Annual Conference on Learning Theory*, pages 1–26. JMLR Workshop and Conference Proceedings, 2011.

Er-Wei Bai, Roberto Tempo, and Hyonyong Cho. Membership set estimators: size, optimal inputs, complexity and relations with least squares. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42 (5):266–277, 1995.

Eli Fogel and Yih-Fang Huang. On the value of information in system identification—bounded noise case. *Automatica*, 18(2):229–238, 1982.

Wataru Kitamura and Yasumasa Fujisaki. Convergence properties of the membership set in the presence of disturbance and parameter uncertainty. *Transactions of the Society of Instrument and Control Engineers*, 39(4):382–387, 2003.

Mario Milanese, John Norton, Hélène Piet-Lahanier, and Éric Walter. *Bounding approaches to system identification*. Springer Science & Business Media, 2013.

Marco Lauricella and Lorenzo Fagiano. Set membership identification of linear systems with guaranteed simulation accuracy. *IEEE Transactions on Automatic Control*, 65(12):5189–5204, 2020.

Jaap Eising, Shenyu Liu, Sonia Martínez, and Jorge Cortés. Using data informativity for online stabilization of unknown switched linear systems. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 8–13. IEEE, 2022.

Mitchell M Livstone and Munther A Dahleh. Asymptotic properties of set membership identification algorithms. *Systems & control letters*, 27(3):145–155, 1996.

Er-Wei Bai, Hyonyong Cho, and Roberto Tempo. Convergence properties of the membership set. *Automatica*, 34(10): 1245–1249, 1998.

Hüseyin Akçay. The size of the membership-set in a probabilistic framework. *Automatica*, 40(2):253–260, 2004.

Wataru Kitamura, Yasumasa Fujisaki, and Er-Wei Bai. The size of the membership set in the presence of disturbance and parameter uncertainty. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 5698–5703. IEEE, 2005.

Xiaonan Lu, Mark Cannon, and Denis Koksal-Rivet. Robust adaptive model predictive control: Performance and parameter estimation. *International Journal of Robust and Nonlinear Control*, 2019.

Johannes Köhler, Elisa Andina, Raffaele Soloperto, Matthias A Müller, and Frank Allgöwer. Linear robust adaptive model predictive control: Computational complexity and conservatism. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 1383–1388. IEEE, 2019.

Stephen L Tu. *Sample complexity bounds for the linear quadratic regulator*. University of California, Berkeley, 2019.

James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.

Sarah Dean, Stephen Tu, Nikolai Matni, and Benjamin Recht. Safely learning to control the constrained linear quadratic regulator. In *2019 American Control Conference (ACC)*, pages 5582–5588. IEEE, 2019a.

Xiaonan Lu and Mark Cannon. Robust adaptive model predictive control with persistent excitation conditions. *Automatica*, 152:110959, 2023.

David Q Mayne, María M Seron, and SV Rakovic. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219–224, 2005.

Sasa V Rakovic, William S Levine, and Behçet Açikmese. Elastic tube model predictive control. In *2016 American Control Conference (ACC)*, pages 3594–3599. IEEE, 2016.

Shaoru Chen, Nikolai Matni, Manfred Morari, and Victor M Preciado. System level synthesis-based robust model predictive control through convex inner approximation. *arXiv preprint arXiv:2111.05509*, 2021.

Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Online least squares estimation with self-normalized processes: An application to bandit problems. *arXiv preprint arXiv:1102.2670*, 2011.

Peter Van Overschee and BL0888 De Moor. *Subspace identification for linear systems: Theory—Implementation—Applications*. Springer Science & Business Media, 2012.

Lennart Ljung. *System identification*. Springer, 1998.

Marco C Campi and Erik Weyer. Finite sample properties of system identification methods. *IEEE Transactions on Automatic Control*, 47(8):1329–1334, 2002.

Mathukumalli Vidyasagar and Rajeeva L Karandikar. A learning theory approach to system identification and stochastic adaptive control. *Probabilistic and randomized methods for design under uncertainty*, pages 265–302, 2006.

Daniel Hsu, Sham M Kakade, and Tong Zhang. Random design analysis of ridge regression. In *Conference on learning theory*, pages 9–1. JMLR Workshop and Conference Proceedings, 2012.

Samet Oymak and Necmiye Ozay. Non-asymptotic identification of lti systems from a single trajectory. In *2019 American control conference (ACC)*, pages 5655–5661. IEEE, 2019.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. On the sample complexity of the linear quadratic regulator. *Foundations of Computational Mathematics*, pages 1–47, 2019b.

Yang Zheng and Na Li. Non-asymptotic identification of linear dynamical systems using multiple trajectories. *IEEE Control Systems Letters*, 5(5):1693–1698, 2020.

Anders Rantzer. Concentration bounds for single parameter adaptive control. In *2018 Annual American Control Conference (ACC)*, pages 1862–1866. IEEE, 2018.

Anastasios Tsiamis, Ingvar Ziemann, Nikolai Matni, and George J Pappas. Statistical learning theory for control: A finite sample perspective. *arXiv preprint arXiv:2209.05423*, 2022.

Zichen Zhao and Qianxiao Li. Adaptive sampling methods for learning dynamical systems. In *Mathematical and Scientific Machine Learning*, pages 335–350. PMLR, 2022.

Henk J van Waarde, Jaap Eising, M Kanat Camlibel, and Harry L Trentelman. The informativity approach to data-driven analysis and control. *arXiv preprint arXiv:2302.10488*, 2023.

Jaap Eising and Jorge Cortes. When sampling works in data-driven control: Informativity for stabilization in continuous time. *arXiv preprint arXiv:2301.10873*, 2023.

Yushuang Liu, Yan Zhao, and Falin Wu. Ellipsoidal state-bounding-based set-membership estimation for linear system with unknown-but-bounded disturbances. *IET Control Theory & Applications*, 10(4):431–442, 2016.

CA Rogers. Covering a sphere with spheres. *Mathematika*, 10(2):157–164, 1963.

Jean-Louis Verger-Gaugry. Covering a ball with smaller equal balls in $\mathbb{R}^n$. *Discrete and Computational Geometry*, 33:143–155, 2005.

Appendix

Roadmap

- Appendix A introduces additional notation used throughout the Appendix.

- Appendix B provides more literature review on LSE and SM, and a more detailed discussion on the technical contributions of this paper.

- Appendix C provides more discussions on examples that satisfy Assumptions 2 and 3.

- Appendix D presents the proof of Theorem 1. In particular, we provide helper lemmas in Appendix D.1 and prove Lemma 1, Lemma 2 in Appendix D.2 and Appendix D.3 respectively.

- Appendix E presents a proof of Corollary 1.

- Appendix F presents a proof of Theorem 2.

A   Additional notations

Let $\mathbb{S}_n(0,1)$ denote the unit sphere in $\mathbb{R}^n$ in l_2 norm, i.e., $\mathbb{S}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Let $\mathbb{S}_{n \times m}(0,1)$ denote the unit sphere in $\mathbb{R}^{n \times m}$ with respect to the Frobenius norm, i.e., $\mathbb{S}_{n \times m}(0,1) = \{M \in \mathbb{R}^{n \times m} : \|M\|_F = 1\}$. Let $\bar{B}_n(0,1)$ denote the closed unit ball in $\mathbb{R}^n$ in l_2 norm, i.e., $\bar{B}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$. Let $\bar{B}_{n \times m}(0,1)$ denote the closed unit ball in $\mathbb{R}^{n \times m}$ in Frobenius norm, i.e., $\bar{B}_{n \times m}(0,1) = \{M \in \mathbb{R}^{n \times m} : \|M\|_F \leq 1\}$. For a matrix $M \in \mathbb{R}^{n \times m}$, $\text{vec}(M)$ is the vectorization of M. Moreover, we define the inverse mapping of $\text{vec}(\cdot)$ as $\text{mat}(\cdot)$, i.e., for a vector $d \in \mathbb{R}^{nm}$, $\text{mat}(d) \in \mathbb{R}^{n \times m}$. Consider a σ-algebra $\mathcal{F}$ and a random variable X, we write $X \in \mathcal{F}$ if X is measurable with respect to $\mathcal{F}$, i.e., for all Borel measurable sets $B \subseteq \mathbb{R}$, we have $X^{-1}(B) \in \mathcal{F}$. We can similarly define $\mathcal{F}$-measurable random matrices and random vectors. Further, consider a polyhedral $\mathbb{D} = \{x : Ax \leq b\}$, we write $\mathbb{D} \in \mathcal{F}$ if matrix A and vector b are measurable with respect to $\mathcal{F}$. Consider two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \succeq B$ if $A - B$ is a positive definite matrix. We define $\min \emptyset = +\infty$. For a set $\mathcal{E}$, let $\mathbb{1}_\mathcal{E}$ denote the indicator function on $\mathcal{E}$. For a vector $x \in \mathbb{R}^n$, we use x^j to denote the jth coordinate of x.

B   More discussions on least square and set membership

System identification studies the problem of estimating the parameters of an unknown dynamical systems from trajectory data. There are two main classes of estimation methods: point estimator such as least square estimation (LSE), and set estimator such as set membership (SM). In the following, we provide more discussions and literature review on LSE and SM. We will also discuss the major technical novelties of this work.

B.1   Least square estimation

For linear dynamical systems $x_{t+1} = A^* x_t + B^* u_t + w_t = \theta^* z_t + w_t$, given a trajectory of data $\{x_t, u_t\}_{t \geq 0}$, least square estimation generates a point estimator that minimizes the following quadratic error Van Overschee and De Moor [2012], Ljung [1998]:

$$\hat{\theta}_{\text{LSE}} = \min_{\hat{\theta}} \sum_{t=0}^{T-1} \|x_{t+1} - \hat{\theta} z_t\|_2^2.$$

Least-square estimation is widely used and its convergence (rate) guarantees have been investigated for a long time. In particular, non-asymptotic convergence rate guarantees of LSE has become increasingly important as these guarantees are the foundations for non-asymptotic performance analysis of learning-based/adaptive control algorithms. Earlier non-asymptotic analysis of LSE focused on the simpler regression model $y_t = \theta^* x_t + w_t$, where x_t and y_t are independent Campi and Weyer [2002], Vidyasagar and Karandikar [2006], Hsu et al. [2012].

Recently, there is one major breakthrough in Simchowitz et al. [2018] that provides LSE's convergence rate analysis for linear dynamical system $x_{t+1} = \theta^* z_t + w_t$, where x_{t+1} and $z_t = [x_t^\top, u_t^\top]^\top$ are correlated. More specifically, Simchowitz et al. [2018] establishes a fundamental property, *block-martingale small-ball (BMSB)*, to analyze LSE under correlated data. BMSB enables a long list of subsequent literature on LSE's non-asymptotic analysis for different types of dynamical systems, e.g., Oymak and Ozay [2019], Dean et al. [2019b], Zheng and Li [2020], Rantzer [2018], Faradonbeh et al. [2018], Wagenmaker and Jamieson [2020], Tsiamis et al. [2022], Zhao and Li [2022], Li et al. [2021].

Though LSE is a point estimator, one can establish confidence region of LSE based on proper statistical assumptions on w_t. The pioneer works on the confidence region of LSE for linear dynamical systems are Abbasi-Yadkori et al. [2011], Abbasi-Yadkori and Szepesvári [2011], which construct ellipsoid confidence regions for LSE. Moreover, the non-asymptotic bounds on estimation errors established in Simchowitz et al. [2018], Dean et al. [2019a] can also be viewed as confidence bounds. Further, the estimation error $\tilde{O}(\frac{\sqrt{n_x + n_z}}{\sqrt{T}})$ has been shown to match the fundamental lower bound for any estimation methods for unbounded disturbances in Simchowitz et al. [2018]. However, these confidence bounds all rely on statistical inequalities, which may result in loose constant factors despite an optimal convergence rate. When applying these confidence bounds to robust control, where the controller is required to satisfy certain stability and constraint satisfaction properties for every possible system in the confidence region, a loose constant factor will result in a larger confidence region and a more conservative control design. Finally, in robust control and many practical applications, the disturbances are usually bounded, and it will be interesting to see how the knowledge of the boundedness will improve the uncertainty set estimation.

B.2 Set membership

Set membership is commonly used in robust control for uncertainty set estimation Milanese and Vicino [1991], Adetola and Guay [2011], Tanaskovic et al. [2013], Bujarbaruah et al. [2020], Zhang et al. [2021], Parsi et al. [2020a,b], Sasfi et al. [2022]. There is a long history of research on SM for both deterministic disturbances, such as Bai et al. [1995], Fogel and Huang [1982], Kitamura and Fujisaki [2003], Milanese et al. [2013], Lauricella and Fagiano [2020], Livstone and Dahleh [1996], and stochastic disturbances, such as Bai et al. [1998, 1995], Kitamura et al. [2005], Akçay [2004], Lu et al. [2019]. For the stochastic disturbances, both convergence and convergence rate analysis have been investigated under the persistent excitation (PE) condition. However, the existing convergence rates are only established for simpler regression problems, $y_t = \theta^* x_t + w_t$, where y_t and x_t are independent Akçay [2004], Bai et al. [1995, 1998], Kitamura et al. [2005].

Recently, Lu et al. [2019] provided an initial attempt to establish the convergence guarantee of SM for linear dynamical systems $x_{t+1} = \theta^* z_t + w_t$ for correlated data x_{t+1} and z_t. However, Lu et al. [2019] assumes that PE holds deterministically, and designs a special control design based on constrained optimization to satisfy PE deterministically. Therefore, the convergence for general control design and the convergence rate analysis remain open questions for correlated data arising from dynamical systems.

In this paper, we establish the convergence rate guarantees of SM on linear dynamical systems under the BMSB conditions in Simchowitz et al. [2018]. Compared with Lu et al. [2019], BMSB condition can be satisfied by adding an i.i.d. random noise to a general class of control designs Li et al. [2021].

Technically, one major challenge of SM analysis compared with the LSE analysis is that the diameter of the membership set does not have an explicit formula, which is in stark contrast with LSE, where the point estimator is the solution to a quadratic program and has explicit form. A common trick to address this issue in the analysis of SM is to connect the diameter bound with the values of disturbances subsequences $\{w_{s_k}\}_{k \geq 0}$: it can be generally shown that a large diameter indicates that a long subsequence of disturbances are far away from the boundary of $\mathcal{W}$. However, existing construction methods of $\{w_{s_k}\}_{k \geq 0}$ will cause the time indices $\{s_k\}_{k \geq 0}$ to *correlate* with the realization of the sequences $\{x_t, u_t, w_t\}_{t \geq 0}$ Akçay [2004], Lu et al. [2019], Bai et al. [1995].[4] Consequently, in the correlated-data scenario and when PE does not hold deterministically, under the existing construction methods in Akçay [2004], Lu et al. [2019], Bai et al. [1995], the probability of $\{w_{s_k}\}_{k \geq 0}$ with correlated time indices *cannot* be bounded by the probability of the *independent* sequence $\{w_t\}_{t \geq 0}$. One major **technical contribution** of this paper is to provide a novel construction of $\{w_{s_k}\}_{k \geq 0}$ based on a sequence of stopping times and establish conditional independence properties despite correlated data and stochastic PE condition (BMSB). More details can be found in Lemma 4 and the proof or Lemma 2.

Though we only consider box constraints for w_t, it is worth mentioning that SM can be applied to much more general forms of disturbances. For example, a common alternative is the ellipsoidal-bounded disturbance where

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[4]In Lu et al. [2019], the correlation between $\{s_k\}_{k \geq 0}$ and $\{x_t, u_t, w_t\}_{t \geq 0}$ is via the PE condition, but Lu et al. [2019] assume deterministic PE to avoid this correlation issue.

$\mathcal{W} := \{w \in \mathcal{R}^{n_x} : w^\top P w \le 1\}$ with positive definite $P \in \mathcal{R}^{n_x \times n_x}$ Bai et al. [1995], van Waarde et al. [2023], Eising and Cortes [2023], Liu et al. [2016] and polytopic-bounded disturbance $\mathcal{W} := \{w \in \mathcal{R}^{n_x} : Gw \le h\}$ for positive definite $G \in \mathcal{R}^{n_x \times n_x}$ and $h \in \mathcal{R}^{n_x}$ Fogel and Huang [1982], Lu et al. [2019], Lu and Cannon [2023]. There are also SM literature assuming bounded energy of the disturbance sequences Bai et al. [1995]. It is an interesting future direction to extend the analysis in this paper to more general disturbance constraints.

A separate but important challenge is that the knowledge of $\mathcal{W}$ is not always available a priori. There is literature discussing the estimation of $\mathcal{W}$ Lauricella and Fagiano [2020], Bai et al. [1998]. We leave for future work how to simultaneously estimate $\mathcal{W}$ and perform non-asymptotic analysis on the size of the membership set.

Further, exact SM involves the intersection of an increasing number of sets, thus causing the computation complexity increases with time t, which can become prohibitive when t is large. There are many methods trying to reduce the computation complexity by approximating the membership sets (see e.g., Livstone and Dahleh [1996], Lu et al. [2019], etc.). It is an exciting future direction to study the diameter bounds of the approximated SM methods.

C   More discussions on Assumptions 2 and 3

C.1   More discussions on Assumption 2

The BMSB condition has been widely used in learning-based control. It has been shown that BMSB can be satisfied in many scenarios. For example, Simchowitz et al. [2018], Tu [2019] showed that linear systems with i.i.d. perturbed linear control policies, i.e., $x_{t+1} = A x_t + B(K x_t + \eta_t) + w_t$,[5] satisfy BMSB if the disturbances w_t and η_t are i.i.d. and follow Gaussian distributions with positive definite covariance matrices. Later, Dean et al. [2019a] showed that $x_{t+1} = A x_t + B(K x_t + \eta_t) + w_t$ can still satisfy BMSB even for non-Gaussian distributions of w_t, η_t, as long as w_t and η_t have independent coordinates and finite fourth moments. Recently, Li et al. [2021] extended the results to linear systems with nonlinear policies, i.e., $x_{t+1} = A x_t + B(\pi_t(x_t) + \eta_t) + w_t$, and showed that BMSB still holds as long as the nonlinear policies π_t generate bounded trajectories of states and control inputs, and w_t, η_t are bounded and follow distributions with certain anti-concentrated properties (a special case is positive definite covariance matrix).

C.2   More discussions on Assumption 3

In this subsection, we provide two example distributions, truncated Gaussian and uniform distributions, and discuss their corresponding $q_w(\epsilon)$ functions. It will be shown that for both distributions below, $q_w(\epsilon) = O(\epsilon)$.

Lemma 5 (Example of uniform distribution). *Consider w_t that follows a uniform distribution on $[-w_{\max}, w_{\max}]^{n_x}$. Then, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}}$.*

Proof. Since $\mathrm{Unif}(\mathcal{W})$ is symmetric, we only need to consider one direction $j = 1$.

$$\mathbb{P}(w^j + w_{\max} \le \epsilon) = \int_{w^1 + w_{\max} \le \epsilon} \int_{w^2, \dots, w^{n_x} \in [-w_{\max}, w_{\max}]} \frac{1}{(2w_{\max})^{n_x}} \mathbb{1}_{(w \in \mathcal{W})} \, \mathrm{d}w$$

$$= \int_{w^1 \le \epsilon - w_{\max}} \frac{1}{2w_{\max}} \mathbb{1}_{(w \in \mathcal{W})} \mathrm{d}w^1 = \frac{\epsilon}{2w_{\max}}$$

Similarly, $\mathbb{P}(w_{\max} - w^1 \le \epsilon) = \int_{w^1 \ge w_{\max} - \epsilon} \frac{1}{2w_{\max}} \mathbb{1}_{(w \in \mathcal{W})} dw^1 = \frac{\epsilon}{2w_{\max}}$. $\qquad\square$

Lemma 6 (Example of truncated Gaussian distribution). *Consider w_t follows a truncated Gaussian distribution on $[-w_{\max}, w_{\max}]^{n_x}$ generated by a Gaussian distribution with zero mean and $\sigma_w I_{n_x}$ covariance matrix. Then, $q_w(\epsilon) = \frac{1}{\min(\sqrt{2\pi}\sigma_w, 2w_{\max})} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})\epsilon$.*

Proof. Since this distribution is symmetric and each coordinate is independent, we only need to consider one direction j. Let X denote a Gaussian distribution with zero mean and σ_w^2 variance. By the definition of truncated Gaussian distributions, we have

$$\mathbb{P}(w^j + w_{\max} \le \epsilon) = \frac{\mathbb{P}(-w_{\max} \le X \le -w_{\max} + \epsilon)}{\mathbb{P}(-w_{\max} \le X \le w_{\max})}$$

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[5]Though we only describe a static linear policy $u_t = K x_t$ here, the results in Simchowitz et al. [2018], Tu [2019], Dean et al. [2019a] hold for dynamic linear policies.

Notice that X/σ_w follows the standard Gaussian distribution, so we can obtain the following bounds.

$$\mathbb{P}(-w_{\max} \leq X \leq -w_{\max} + \epsilon) = \int_{-w_{\max}/\sigma_w}^{(-w_{\max}+\epsilon)/\sigma_w} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) \, \mathrm{d}z$$

$$\geq \frac{1}{\sqrt{2\pi}} \exp(-w_{\max}^2/(2\sigma_w^2)) \frac{\epsilon}{\sigma_w}$$

and

$$\mathbb{P}(-w_{\max} \leq X \leq w_{\max}) = \int_{-w_{\max}/\sigma_w}^{w_{\max}/\sigma_w} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) \, \mathrm{d}z$$

$$\leq \min(1, \frac{1}{\sqrt{2\pi}} \frac{2w_{\max}}{\sigma_w})$$

Therefore, we obtain

$$\mathbb{P}(w^j + w_{\max} \leq \epsilon) = \frac{\mathbb{P}(-w_{\max} \leq X \leq -w_{\max} + \epsilon)}{\mathbb{P}(-w_{\max} \leq X \leq w_{\max})}$$

$$\geq \max(\frac{1}{\sqrt{2\pi}} \exp(-w_{\max}^2/\sigma_w^2) \frac{\epsilon}{\sigma_w}, \frac{\epsilon}{2w_{\max}} \exp(\frac{-w_{\max}^2}{2\sigma_w^2}))$$

$$= \frac{1}{\min(\sqrt{2\pi}\sigma_w, 2w_{\max})} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})\epsilon$$

Finally, $\mathbb{P}(w_{\max} - w^1 \leq \epsilon)$ can be bounded similarly. $\qquad\square$

D   Proof of Theorem 1

The section provides more details for the proof of Theorem 1. In particular, we first provide technical lemmas for set discretization, then prove Lemma 1 and Lemma 2 respectively. The proof of Theorem 1 follows naturally by combining Lemma 1 and Lemma 2.

D.1   Technical lemmas: set discretization

In this subsection, we provide technical results of finite-ball covering. The results are classical and are based on Rogers [1963] Verger-Gaugry [2005].

Theorem 3 (Ball covering Rogers [1963] Verger-Gaugry [2005]). *Consider a closed ball $\bar{\mathbb{B}}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ in l_2 norm. Considering covering this ball $\bar{\mathbb{B}}_n(0,1)$ with smaller closed balls $\bar{\mathbb{B}}_n(z,\epsilon)$ for $z \in \mathbb{R}^n$. Let $v_{\epsilon,n}$ denote the minimal number of smaller balls needed to cover $\bar{\mathbb{B}}_n(0,1)$. Then, we have*

$$v_{\epsilon,n} \leq \begin{cases} \tilde{O}(n) \left(\frac{1}{\epsilon}\right)^n, & \text{if } \epsilon \leq \frac{1}{n}, \\ \tilde{O}(n^{5/2}) \left(\frac{1}{\epsilon}\right)^n, & \text{if } \epsilon \leq 1. \end{cases}$$

Corollary 4. *There exists a finite set $\mathcal{M}' = \{\lambda_1, \ldots, \lambda_{v_\lambda}\} \subseteq \mathbb{S}_{n_z}(0,1)$ such that for any $\lambda \in \mathbb{R}^{n_z}$ with $\|\lambda\|_2 = 1$, there exists $\lambda_i \in \mathcal{M}'$ such that $\|\lambda - \lambda_i\|_2 \leq 2\epsilon_\lambda$. Further, $v_\lambda \leq \tilde{O}(n_z^{2.5}) \left(\frac{1}{\epsilon_\lambda}\right)^{n_z}$ if $\epsilon_\lambda \leq 1$.*

In the following, we consider $\epsilon_\lambda = \min(\sigma_z^2 p_z^2/(64b_z^2), 1)$. Hence,

$$v_\lambda \leq \tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}). \tag{6}$$

Next, we apply Theorem 3 to a space of matrices.

Lemma 7. *There exists a finite set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\} \subseteq \mathbb{S}_{n_x \times n_z}(0,1)$ such that for any $\gamma \in \mathbb{R}^{n_x \times n_z}$ and $\|\gamma\|_F = 1$, there exists $\gamma_i \in \mathcal{M}$ such that $\|\gamma - \gamma_i\|_F \leq 2\epsilon_\gamma$. Further, $v_\gamma \leq \tilde{O}(n_x^{2.5} n_z^{2.5}) \left(\frac{1}{\epsilon_\gamma}\right)^{n_z n_x}$ if $\epsilon_\gamma \leq 1$.*

In the following, we consider $\epsilon_\gamma = \min(\frac{a_1}{4b_z \sqrt{n_x}}, 1)$.

Proof. The proof is basically by mapping the matrices to vectors based on matrix vectorization, then mapping the vectors back to matrices. These two mappings are isomorphism.

Specifically, consider a closed unit ball in $\mathbb{R}^{n_x n_z}$. There exist $v_{\epsilon, n_x n_z}$ smaller closed balls to cover it, denoted by $\mathbb{B}_1, \ldots, \mathbb{B}_{v_{\epsilon, n_x n_z}}$. Consider the non-empty sets from $\mathbb{B}_1 \cap \mathbb{S}_{n_x n_z}(0, 1), \ldots, \mathbb{B}_{v_{\epsilon, n_x n_z}} \cap \mathbb{S}_{n_x n_z}(0, 1)$. For any $1 \leq i \leq v_{\epsilon, n_x n_z}$, if $\mathbb{B}_i \cap \mathbb{S}_{n_x n_z}(0, 1) \neq \emptyset$, select a point $\text{vec}(\gamma) \in \mathbb{B}_i \cap \mathbb{S}_{n_x n_z}(0, 1)$. Notice that $\|\text{vec}(\gamma)\|_2 = 1$. In this way, we construct a finite sequence $\{\text{vec}(\gamma_1), \ldots, \text{vec}(\gamma_{v_\gamma})\}$ where $v_\gamma \leq v_{\epsilon_\gamma, n_x n_z}$.[6]

For any $\gamma \in \mathbb{R}^{n_x \times n_z}$, we have $\text{vec}(\gamma) \in \mathbb{R}^{n_x n_z}$ and $\|\text{vec}(\gamma)\|_2 = 1$. Hence, there exists $1 \leq i \leq v_\gamma$ such that $\text{vec}(\gamma) \in \mathbb{B}_i \cap \mathbb{S}_{n_x n_z}(0, 1)$. Hence, $\|\text{vec}(\gamma) - \text{vec}(\gamma_i)\|_2 \leq 2\epsilon_\gamma$. Moreover, $\|\gamma_i\|_F = \|\text{vec}(\gamma_i)\|_2 = 1$. Therefore, $\|\gamma_i - \gamma\|_F \leq 2\epsilon_\gamma$. So the set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$ satisfies our requirement. □

D.2 Proof of Lemma 1

The proof leverages the following results related to Proposition 2.5 in Simchowitz et al. [2018].

Definition 4 (BMSB for scalar sequence)**.** *Let $\{Z_t\}_{t \geq 1}$ be an $\{\mathcal{F}_t\}_{t \geq 1}$-adapted random process taking values in $\mathbb{R}$. We say $\{Z_t\}_{t \geq 1}$ satisfies the (k, v, p)-block martingale small-ball (BMSB) condition if, for any $j \geq 0$, one has $\frac{1}{k} \sum_{i=1}^{k} \mathbb{P}(|Z_{j+i}| \geq v \mid \mathcal{F}_t) \geq p$ almost surely.*

Theorem 4 (Proposition 2.5 in Simchowitz et al. [2018] when $k = 1$)**.** *Let $\{Z_t\}_{t \geq 0}$ be a scalar process that is $(1, v, p)$-BMSB and $Z_t \in \mathcal{F}_t$ for $t \geq 0$, where Z_0 is given. Then*

$$\mathbb{P}(\sum_{t=1}^{T} Z_t^2 \leq v^2 p^2 T / 8) \leq \exp(-T p^2 / 8)$$

Corollary 5 (Implications of Proposition 2.5 in Simchowitz et al. [2018] conditioning on $\mathcal{F}_t$)**.** *Let $\{Z_t\}_{t \geq 0}$ be a scalar process that is $(1, v, p)$-BMSB and $Z_t \in \mathcal{F}_t$ for $t \geq 0$, where Z_0 is given. Then*

$$\mathbb{P}(\sum_{t=s+1}^{T} Z_t^2 \leq v^2 p^2 (T - s) / 8 \mid \mathcal{F}_s) \leq \exp(-(T - s) p^2 / 8), \quad \forall s \geq 0$$

Proof. Consider $\{Y_t\}_{t \geq 0}$ where $Y_t = Z_{t+s}$ for $t \geq 0$. Conditioning on $\mathcal{F}_s$, Y_0 is given. Further, conditioning on $\mathcal{F}_s$, $\{Y_t\}$ is BMSB because

$$\mathbb{P}(|Y_{t+1}| \geq v \mid \mathcal{F}_t^y \cap \mathcal{F}_s) = \mathbb{P}(|Z_{t+1+s}| \geq v \mid \mathcal{F}_{t+s}) \geq p.$$

Hence, we can apply Theorem 4 on $\{Y_t\}$ for $t = 1, \ldots, T - s$. □

Lemma 8. *For any $\lambda \in \mathbb{R}^{n_z}$ such that $\|\lambda\|_2 = 1$, for any $k \geq 0$, $m \geq 1$, we have*

$$\mathbb{P}(\sum_{i=1}^{m} \lambda^\top z_{km+i} z_{km+i}^\top \lambda \leq \sigma_z^2 p_z^2 m / 8 \mid \mathcal{F}_{km}) \leq \exp(-m p_z^2 / 8)$$

Proof. Firstly, by Assumption 2, $\{z_t\}$ is $\mathcal{F}_t$ adapted and $(1, \sigma_z^2 I_{n_z}, p_z)$-BMSB. Therefore, for any λ with $\|\lambda\|_2 = 1$, $\lambda^\top z_t$ is $(1, \sigma_z, p_z)$-BMSB. Therefore, by Corollary 5, we have

$$\mathbb{P}(\sum_{t=km+1}^{km+m} \lambda^\top z_t z_t^\top \lambda \leq \sigma_z^2 p_z^2 m / 8 \mid \mathcal{F}_{km}) \leq \exp(-m p_z^2 / 8), \quad \forall k \geq 0, m \geq 1$$

This completes the proof. □

Lemma 9 (Definition of $H_{i,k}$)**.** *Consider $\mathcal{M}' = \{\lambda_1, \ldots, \lambda_{v_\lambda}\}$ as defined in Corollary 4. For any $m \geq 1$, define events*

$$H_{i,k} = \left\{ \sum_{t=km+1}^{km+m} \lambda_i^\top z_t z_t^\top \lambda_i > \sigma_z^2 p_z^2 m / 8 \right\}, \quad \forall k \geq 0, 1 \leq i \leq v_\lambda$$

Then $\mathbb{P}(H_{i,k}^c \mid \mathcal{F}_{km}) \leq \exp(-m p_z^2 / 8)), \quad \forall k \geq 0, 1 \leq i \leq v_\lambda$.

---

[6] Here, without loss of generality, we consider $\mathbb{B}_1 \cap \mathbb{S}_{n_x n_z}(0, 1), \ldots, \mathbb{B}_{v_\gamma} \cap \mathbb{S}_{n_x n_z}(0, 1)$ are not empty.

Proof. For any $\lambda_i \in \mathcal{M}'$, $\|\lambda_i\|_2 = 1$, so by Lemma 8, we have $\mathbb{P}(H_{i,k}^c \mid \mathcal{F}_{km}) \leq \exp(-mp_z^2/8))$. $\qquad \square$

Lemma 10 (Probability of intersections of $H_{i,k}$). $\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$.

Proof. We can establish the following:

$$\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) = 1 - \mathbb{P}(\bigcup_{i=1}^{v_\lambda} H_{i,k}^c \mid F_{km}) \geq 1 - \sum_{i=1}^{v_\lambda} \mathbb{P}(H_{i,k}^c \mid F_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$$

where the last inequality is by Lemma 9. $\qquad \square$

Lemma 11 (Probability of PE in one segment). *For any $m \geq 1$, for any $k \geq 0$, we have*

$$\mathbb{P}(\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16)I_{n_z} \mid \mathcal{F}_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$$

Proof. By Corollary 4, for any λ such that $\|\lambda\|_2 = 1$, there exists $\lambda_i \in \mathcal{M}'$ such that $\|\lambda - \lambda'\|_2 \leq 2\epsilon_\lambda$. Suppose for this λ_i, we have $\sum_{t=km+1}^{km+m} \lambda_i^\top z_t z_t^\top \lambda_i > \sigma_z^2 p_z^2 m/8$. Then

$$
\begin{aligned}
\sum_{t=km+1}^{km+m} \lambda^\top z_t z_t^\top \lambda &= \sum_{t=km+1}^{km+m} \lambda_i^\top z_t z_t^\top \lambda_i + \sum_{t=km+1}^{km+m} (\lambda + \lambda_i)^\top z_t z_t^\top (\lambda - \lambda_i) \\
&> \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} (\lambda + \lambda_i)^\top z_t z_t^\top (\lambda_i - \lambda) \\
&\geq \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} \|\lambda + \lambda_i\|_2 \|z_t\|_2^2 \|\lambda_i - \lambda\|_2 \\
&\overset{(a)}{\geq} \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} 4b_z^2 \epsilon_\lambda \\
&= \sigma_z^2 p_z^2 m/8 - 4b_z^2 \epsilon_\lambda m \overset{(b)}{\geq} \sigma_z^2 p_z^2 m/16
\end{aligned}
$$

where (a) is by Assumption 2 and $\|\lambda - \lambda'\|_2 \leq 2\epsilon_\lambda$, and (b) is by $\epsilon_\lambda \leq \sigma_z^2 p_z^2/(64b_z^2)$.

Therefore, together with Lemma 10, we complete the proof.

$$
\begin{aligned}
\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) &\leq \mathbb{P}(\forall \|\lambda\|_2 = 1, \sum_{t=km+1}^{km+m} \lambda^\top z_t z_t^\top \lambda > \sigma_z^2 p_z^2 m/16 \mid F_{km}) \\
&= \mathbb{P}(\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16)I_{n_z} \mid \mathcal{F}_{km})
\end{aligned}
$$

$\qquad \square$

Proof of Lemma 1. Recall that $\mathcal{E}_2 = \{\frac{1}{m}\sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}, \forall 0 \leq k \leq \lceil (T-1)/m \rceil - 1\}$. Hence

$$\mathcal{E}_2 = \bigcap_{k=0}^{(T-1)/m-1} \{\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16)I_{n_z}\}$$

where $a_1 = \sigma_z p_z/4$.

For notational simplicity, we define $\bar{H}_k = \{\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16)I_{n_z}\}$. So

$$\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} \bar{H}_k) = 1 - \mathbb{P}(\bigcup_{k=0}^{(T-1)/m-1} \bar{H}_k^c)$$

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$$\geq 1 - \sum_{k=0}^{(T-1)/m-1} \mathbb{P}(\bar{H}_k^c) = 1 - \sum_{k=0}^{(T-1)/m-1} (1 - \mathbb{P}(\bar{H}_k))$$

By Lemma 11, we have $\mathbb{P}(\bar{H}_k \mid \mathcal{F}_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$. Therefore, $1 - \mathbb{P}(\bar{H}_k \mid \mathcal{F}_{km}) \leq v_\lambda \exp(-mp_z^2/8))$. Therefore,

$$\begin{aligned} \mathbb{P}(\mathcal{E}_2) &\geq 1 - ((T-1)/m)v_\lambda \exp(-mp_z^2/8)) \\ &\geq 1 - ((T-1)/m)\tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)) \\ &\geq 1 - \frac{T}{m}\tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)), \end{aligned}$$

where in the second inequality we used (6). Finally

$$\mathbb{P}(\mathcal{E}_2^c) = 1 - \mathbb{P}(\mathcal{E}_2) \leq \frac{T}{m}\tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)).$$

$\square$

D.3 Proof of Lemma 2

This proof takes four major steps:

(i) Define $b_{i,t}, j_{i,t}, L_{i,k}$.

(ii) Provide a formal definition of $\mathcal{E}_{1,k}$ based on $b_{i,t}, j_{i,t}, L_{i,k}$ and prove a formal version of Lemma 3.

(iii) Prove Lemma 4.

(iv) Prove Lemma 2 by the formal version of Lemma 3 and Lemma 4.

It is worth mentioning that the formal definition of $\mathcal{E}_{1,k}$ is slightly different from the definition in Lemma 3, but we still have $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,k} \cap \mathcal{E}_2)$, which is the key property that will be used in the proof of Lemma 2.

D.3.1 Step (i): Definitions of $b_{i,t}, j_{i,t}, L_{i,k}$.

Recall the discretization of $\mathbb{S}_{n_x \times n_z}(0,1)$ in Lemma 7, which generates the set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$. We are going to define $b_{i,t}, j_{i,t}, L_{i,k}$ for $\gamma_i \in \mathcal{M}$ for each $1 \leq i \leq v_\gamma$. Notice that $\mathcal{M}$ is a deterministic set of matrices.

Lemma 12 (Definition of $b_{i,t}, j_{i,t}$). *For any $\gamma_i \in \mathcal{M}$, any $0 \leq t \leq T$, there exist $b_{i,t} \in \{-1, 1\}$ and $1 \leq j_{i,t} \leq n_x$ such that $b_{i,t}, j_{i,t} \in \mathcal{F}(z_t) \subseteq \mathcal{F}_t$ and*

$$\|\gamma_i z_t\|_\infty = b_{i,t}(\gamma_i z_t)^{j_{i,t}}.$$

Note that one way to determine $b_{i,t}, j_{i,t}$ from z_t is by the following: first pick the smallest j such that $|(\gamma_i z_t)^j| = \|\gamma_i z_t\|_\infty$, then let $b_{i,t} = \text{sgn}((\gamma_i z_t)^j)$, where $\text{sgn}(\cdot)$ denotes the sign of a scalar argument.

Proof. For any $\gamma_i \in \mathcal{M}$, any $0 \leq t \leq T$, we have

$$\|\gamma_i z_t\|_\infty = \max_{1 \leq j \leq n_x} \max_{b \in \{-1,1\}} b(\gamma_i z_t)^j$$

Hence, there exist $b_{i,t}, j_{i,t}$ such that $\|\gamma_i z_t\|_\infty = b_{i,t}(\gamma_i z_t)^{j_{i,t}}$. Further, $b_{i,t}, j_{i,t}$ only depend on γ_i and z_t, so they are $\mathcal{F}(z_t)$-measurable, and $\mathcal{F}(z_t) \subseteq \mathcal{F}_t$. $\square$

Lemma 13 (Definition of stopping times $L_{i,k}$). *Let $\eta = \frac{a_1}{\sqrt{n_x}}$. For any $\gamma_i \in \mathcal{M}$, any $0 \leq k \leq (T-1)/m - 1$, we can define a random time index $1 \leq L_{i,k} \leq m+1$ by*

$$L_{i,k} = \min(m+1, \min\{l \geq 1 : \|\gamma_i z_{km+l}\|_\infty \geq \eta\}).$$

Then, we have $1 \leq L_{i,k} \leq m+1$. Further, for any $1 \leq l \leq m$, $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$, and $\{L_{i,k} = m+1\} \in \mathcal{F}_{km+m} \subseteq \mathcal{F}_{km+m+1}$. In other words, $L_{i,k}$ is a stopping time with respect to filtration $\{\mathcal{F}_{km+l}\}_{l \geq 1}$.

Proof. For any i and any k, it is straightforward to see that $L_{i,k}$ is well-defined and $1 \leq L_{i,k} \leq m + 1$.

When $L_{i,k} = l \leq m$, this is equivalent with $\|\gamma_i z_{km+l}\|_\infty \geq \eta$ but $\|\gamma_i z_{km+s}\| < \eta$ for $1 \leq s < l$. Notice that this event is only determined by $z_{km+l}, \ldots, z_{km+1}$, so $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$.

When $L_{i,k} = m + 1$, this is equivalent with $\|\gamma_i z_{km+s}\| < \eta$ for $1 \leq s \leq m$. Notice that this event is only determined by $z_{km+m}, \ldots, z_{km+1}$, so $\{L_{i,k} = m + 1\} \in \mathcal{F}_{km+m}$.

Therefore, by definition, $L_{i,k}$ is a stopping time with respect to filtration $\{\mathcal{F}_{km+l}\}_{l \geq 1}$. $\qquad\square$

D.3.2 Step (ii): a formal version of Lemma 3 and its proof

Lemma 14 (Discretization of $\mathcal{E}_1 \cap \mathcal{E}_2$ (Formal version of Lemma 3))**.** *Let* $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$ *be an* ϵ_γ*-net of* $\{\gamma : \|\gamma\|_F = 1\}$ *as defined in Lemma 7, where* $\epsilon_\gamma = \min(\frac{a_1}{4b_z\sqrt{n_x}}, 1)$, $v_\gamma = \tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}$, *and* $a_4 = \max(\frac{4b_z\sqrt{n_x}}{a_1}, 1)$. *Define*

$$\mathcal{E}_{1,i} = \{\exists \gamma \in \Gamma_T, \text{ s.t. } b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1 \delta}{4\sqrt{n_x}}, \forall k \geq 0\}.$$

Then, we have

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2).$$

The rest of this subsubsection is dedicated to the proof of Lemma 14. As an overview: firstly, we will discuss the implications of $\mathcal{E}_2$ on $\gamma_i \in \mathcal{M}$. Then, we discuss the implications of $\mathcal{E}_2$ on any γ. Lastly, we prove Lemma 14 by combining the implications of $\mathcal{E}_2$ on any γ and $\|\gamma\|_F \geq \delta/2$.

Lemma 15 (The implication of $\mathcal{E}_2$ on γ_i)**.** *If* $\mathcal{E}_2$ *happens, then for any* $\gamma_i \in \mathcal{M}$*, any* $0 \leq k \leq (T-1)/m - 1$*, we have*

$$\max_{1 \leq s \leq m} \|\gamma_i z_{km+s}\|_\infty \geq \frac{a_1}{\sqrt{n_x}}.$$

Therefore, almost surely, we have $1 \leq L_{i,k} \leq m$ *and*

$$b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{\sqrt{n_x}}.$$

Proof. If $\mathcal{E}_2$ happens, then by definition, we have

$$\frac{1}{m} \sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z},$$

for all $0 \leq k \leq (T-1)/m - 1$.

Now, for any $\gamma_i \in \mathcal{M}$, we have that

$$\frac{1}{m} \sum_{s=1}^{m} \gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top \succeq a_1^2 \gamma_i \gamma_i^\top. \tag{7}$$

Therefore, by taking trace at each side of (7), we obtain

$$\frac{1}{m} \sum_{s=1}^{m} \text{tr}(\gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top) \geq a_1^2 \text{tr}(\gamma_i \gamma_i^\top) \tag{8}$$

Since $\gamma_i \in \mathbb{S}_{n_x \times n_z}(0, 1)$, we have $\|\gamma_i\|_F = 1$, so $\text{tr}(\gamma_i \gamma_i^\top) = \text{tr}(\gamma_i^\top \gamma_i) = \|\gamma_i\|_F^2 = 1$. Further, we have

$$\text{tr}(\gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top) = \text{tr}(z_{km+s}^\top \gamma_i^\top \gamma_i z_{km+s}) = z_{km+s}^\top \gamma_i^\top \gamma_i z_{km+s} = \|\gamma_i z_{km+s}\|_2^2.$$

Consequently, we have

$$\frac{1}{m} \sum_{s=1}^{m} \|\gamma_i z_{km+s}\|_2^2 \geq a_1^2$$

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for all k.

By the pigeonhole principle, we have that

$$\max_{1 \leq s \leq m} \|\gamma_i z_{km+s}\|_2^2 \geq a_1^2.$$

This is equivalent with $\max_{1 \leq s \leq m} \|\gamma_i z_{km+s}\|_2 \geq a_1$.

Notice that $\|\gamma_i z_{km+s}\|_2 \leq \sqrt{n_x} \|\gamma_i z_{km+s}\|_\infty$, so $\max_{1 \leq s \leq m} \sqrt{n_x} \|\gamma_i z_{km+s}\|_\infty \geq a_1$, which completes the proof of the first inequality in the lemma statement.

Next, we prove the second inequality in the lemma statement. Notice that by the definition of $L_{i,k}$ in Lemma 13 and by $\eta = \frac{a_1}{\sqrt{n_x}}$, we have $1 \leq L_{i,k} \leq m$ and $\|\gamma_i z_{km+L_{i,k}}\|_\infty \geq \frac{a_1}{\sqrt{n_x}}$ for all k. Further, by Lemma 12, we have $\|\gamma_i z_{km+L_{i,k}}\|_\infty = b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}}$ almost surely. Hence, we have $b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{\sqrt{n_x}}$, which completes the proof.

$\square$

Lemma 16 (The implication of $\mathcal{E}_2$ on γz_t). *If $\mathcal{E}_2$ happens, then for any $\gamma \in \mathbb{R}^{n_x \times n_z}$, there exists $1 \leq i \leq v_\gamma$, such that*

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F,$$

for all $0 \leq k \leq (T-1)/m - 1$.

Proof. Firstly, when $\gamma = 0$, the inequality holds because both sides are 0.

Next, when $\gamma \neq 0$, it suffices to prove $b_{i,km+L_{i,k}}(\frac{\gamma}{\|\gamma\|_F} z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}$. Therefore, we will only consider $\gamma \in \mathbb{S}_{n_x \times n_z}(0,1)$. By Lemma 7, there exists $\gamma_i \in \mathcal{M}$ such that $\|\gamma - \gamma_i\|_F \leq 2\epsilon_\gamma = \min(\frac{a_1}{2b_z \sqrt{n_x}}, 2)$. Notice that by Lemma 15, if $\mathcal{E}_2$ happens, for all k, we have

$$b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{\sqrt{n_x}}.$$

Therefore,

$$
\begin{aligned}
b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} &= b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \\
&\quad - b_{i,km+L_{i,k}}((\gamma_i - \gamma) z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \\
&\geq \frac{a_1}{\sqrt{n_x}} - |b_{i,km+L_{i,k}}((\gamma_i - \gamma) z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}}| \\
&\geq \frac{a_1}{\sqrt{n_x}} - \|(\gamma_i - \gamma) z_{km+L_{i,k}}\|_2 \\
&\geq \frac{a_1}{\sqrt{n_x}} - \|\gamma_i - \gamma\|_2 \|z_{km+L_{i,k}}\|_2 \\
&\geq \frac{a_1}{\sqrt{n_x}} - 2\epsilon_\gamma b_z \geq \frac{a_1}{2\sqrt{n_x}}
\end{aligned}
$$

$\square$

Proof of Lemma 14. By Lemma 16, under $\mathcal{E}_2$, for any $\gamma \in \mathbb{R}^{n_x \times n_z}$, there exists $1 \leq i \leq v_\gamma$, such that

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F,$$

for all $0 \leq k \leq (T-1)/m - 1$. Therefore, if $\mathcal{E}_1 \cap \mathcal{E}_2$ happens, there exists $\gamma \in \Gamma_T$ and a corresponding i, such that

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F \geq \frac{a_1 \delta}{4\sqrt{n_x}}.$$

Therefore,

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcup_{i=1}^{v_\gamma} \mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2),$$

which completes the proof.

$\square$

D.3.3 Proof of Lemma 4

Notice that Lemma 4 states two inequalities: in the following, we will first prove the first inequality $\mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \mathbb{P}(\cap_{k=0}^{(T-1)/m-1} G_{i,k})$, then prove the second inequality on $\mathbb{P}(G_{i,k} \mid \cap_{k'=0}^{k-1} G_{i,k'})$.

Lemma 17 (Bound $\mathcal{E}_{1,i} \cap \mathcal{E}_2$ by $G_{i,k}$). *Under the conditions in Lemma 4, for any $1 \leq i \leq v_\gamma$, we have*

$$\mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k}).$$

Proof. Firstly, for any $\gamma \in \Gamma_T$, we have $\|w_t - \gamma z_t\|_\infty \leq w_{\max}$ for all $t \geq 0$. This suggests that, for any $1 \leq j \leq n_x$, we have

$$-w_{\max} \leq w_t^j - (\gamma z_t)^j \leq w_{\max}.$$

Hence, we have $b(\gamma z_t)^j \leq b w_t^j + w_{\max}$ for any $b \in \{-1, 1\}$, $1 \leq j \leq n_x$, and $t \geq 0$.

Next, by $\mathcal{E}_{1,i}$, there exists $\gamma \in \Gamma_T$ such that $b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ for all $k \geq 0$. Therefore, $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ for all k.

Finally, $\mathcal{E}_{1,i} \cap \mathcal{E}_2$ implies that $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ and $\frac{1}{m} \sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}$ for all k, which is $\bigcap_k G_{i,k}$ by the definition of $G_{i,k}$. $\qquad\square$

Lemma 18 (Bound on $\mathbb{P}(G_{i,k} \mid \cap_{k'=0}^{k-1} G_{i,k'})$). *Under the conditions in Lemma 4, for any $1 \leq i \leq v_\gamma$ and any $k \geq 0$, we have*

$$\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}).$$

Proof. Firstly, notice that when $\frac{1}{m} \sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}$, we have $1 \leq L_{i,k} \leq m$ by the proof of Lemma 15. Therefore, we have

$$\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq \mathbb{P}(b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, 1 \leq L_{i,k} \leq m \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$\leq \sum_{l=1}^{m} \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$\leq \sum_{l=1}^{m} \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}) \mathbb{P}(L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$\overset{(a)}{\leq} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})) \sum_{l=1}^{m} \mathbb{P}(L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$\leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})$$

The inequality (c) is proved in the following:

$$\mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$= \int_{v_{0:km+l}} \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}) \mathrm{d}v_{0:km+l}$$

$$= \int_{v_{0:km+l} \in S_{km+l}} \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid w_{0:km+l} = v_{0:km+l})$$

$$\times \mathbb{P}(w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}) \mathrm{d}v_{0:km+l}$$

$$\overset{(b)}{\leq} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})) \int_{v_{0:km+l} \in S_{km+l}} \mathbb{P}(w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}) \mathrm{d}v_{0:km+l}$$

$$= 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}),$$

where we define a shorthand notation $w_{0:km+l} = (w_0, \ldots, w_{km+l-1})$, and we use $v_{0:km+l}$ to denote a realization of $w_{0:km+l}$, then we define the set of values of $w_{0:km+l}$ as S_{km+l} such that $L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}$ holds. Notice that $L_{i,k} = l$ can be determined by a set of values of $w_{0:km+l}$ because $L_{i,k}$ is a stopping time of $\{F_{km+l}\}_{l \geq 1}$ and thus $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$. The inequality (b) above is because of the following: firstly, notice that $b_{i,km+l}, j_{i,km+l} \in \mathcal{F}_{km+l}$, so $b_{i,km+l}, j_{i,km+l}$ are deterministic values when $w_{0:km+l} = v_{0:km+l}$. Further, since w_{km+l} is independent of $w_{0:km+l}$, we have $\mathbb{P}(w_{max} + bw_{km+l}^j \geq \epsilon \mid w_{0:km+l} = v_{0:km+l}) \leq 1 - q_w(\epsilon)$ for any deterministic b, j and any $\epsilon > 0$ by Assumption 3. Hence, we have $\mathbb{P}(b_{i,km+l}w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid w_{0:km+l} = v_{0:km+l}) \leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})$. $\square$

D.3.4 Proof of Lemma 2

The proof is by leveraging Lemma 14 and Lemma 4.

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2)$$

$$\leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k})$$

$$= \sum_{i=1}^{v_\gamma} \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0}) \cdots \mathbb{P}(G_{i,(T-1)/m-1} \mid \bigcap_{k=0}^{(T-1)/m-2} G_{i,k})$$

$$\leq \sum_{i=1}^{v_\gamma} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{(T-1)/m}$$

$$\leq \tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{(T-1)/m}.$$

E   Proof of Corollary 1

The proof involves two parts. Firstly, we will show that Term 1 $\leq \epsilon$ under our choice of m. Secondly, we will let Term 2 $= \epsilon$, then we will show $\delta \leq \tilde{O}(n_x^{1.5} n_z^2 / T)$, which completes the proof.

Step 1: show Term 1 $\leq \epsilon$. Notice that when $m \geq \frac{1}{a_3} O(\log T + 2.5 \log n_z + n_z \log a_2 + \log(1/\epsilon)) = O(n_z + \log T + \log(1/\epsilon))$, we have $T\tilde{O}(n_z^{2.5}) a_2^{n_z} \exp(-a_3 m) \leq \epsilon$. Since $m \geq 1$, we obtain Term 1 $\leq \epsilon$.

Step 2: let Term 2 $= \epsilon$ and show $\delta \leq \tilde{O}(n_x^{1.5} n_z^2 / T)$. Let Term 2 $= \epsilon$, then we have $(1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{T/m} = \frac{\epsilon}{\tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}}$. Then, we obtain $(1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})) = \left(\frac{\epsilon}{\tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}}\right)^{m/T}$, which is equivalent with

$$q_w(\frac{a_1\delta}{4\sqrt{n_x}}) = 1 - \left(\frac{\epsilon}{\tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}}\right)^{m/T}.$$

When $q_w(\frac{a_1\delta}{4\sqrt{n_x}}) = O(\frac{a_1\delta}{4\sqrt{n_x}})$, we obtain

$$\delta = O(\frac{4\sqrt{n_x}}{a_1})\left(1 - \left(\frac{\epsilon}{\tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}}\right)^{m/T}\right)$$

$$\leq O(\frac{-4\sqrt{n}_x}{a_1})\log\left(\left(\frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_x n_z}}\right)^{m/T}\right)$$

$$= O(\frac{4\sqrt{n}_x m}{a_1 T})(\log(1/\epsilon) + n_x n_z + \log(n_x n_z))$$

$$= \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right).$$

Step 3: prove Corollary 1. By leveraging the bounds above and Theorem 1, we have $\mathbb{P}(\mathrm{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right)) \geq \mathbb{P}(\mathrm{diam}(\Theta_T) \leq \delta) \geq 1 - 2\epsilon$.

Since $\theta^* \in \Theta_T$ by definition, for any $\hat{\theta}_T \in \Theta_T$, we have $\|\hat{\theta}_T - \theta^*\|_F \leq \mathrm{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right)$ with probability at least $1 - 2\epsilon$.

F   Proof of Theorem 2

Specifically, we define $\epsilon_0 = \frac{4\sqrt{n}_x}{a_1}(\hat{w}_{\max} - w_{\max})$.

The proof is similar to the proof of Theorem 1. Firstly, we define $\hat{\Gamma}_T$ as a translation of the set $\hat{\Theta}_T$:

$$\hat{\Gamma}_t = \bigcap_{s=0}^{t-1}\{\gamma : \|w_s - \gamma z_s\|_\infty \leq \hat{w}_{\max}\}, \quad \forall t \geq 0. \tag{9}$$

Notice that

$$\hat{\Theta}_T = \theta^* + \hat{\Gamma}_T$$

by considering $\gamma = \hat{\theta} - \theta^*$. Therefore, we can upper bound our goal event $\{\mathrm{diam}(\hat{\Theta}_T) > \delta + \epsilon_0\}$ by the event $\mathcal{E}_3$ defined below.

$$\mathbb{P}(\mathrm{diam}(\hat{\Theta}_T) > \delta + \epsilon_0) \leq \mathbb{P}(\mathcal{E}_3), \text{ where } \mathcal{E}_3 := \{\exists\,\gamma \in \hat{\Gamma}_T, \text{ s.t. } \|\gamma\|_F \geq \frac{\delta + \epsilon_0}{2}\}. \tag{10}$$

Next, notice that

$$\mathbb{P}(\mathrm{diam}(\hat{\Theta}_T) > \delta + \epsilon_0) \leq \mathbb{P}(\mathcal{E}_3) \leq \mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2) + \mathbb{P}(\mathcal{E}_2^c)$$

By Lemma 1, we have already shown $\mathbb{P}(\mathcal{E}_2^c) \leq$ Term 1. So we only need to discuss $\mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2)$.

Lemma 19.

$$\mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2) \leq \text{Term 2}$$

Proof. Firstly, define

$$\mathcal{E}_{3,i} = \{\exists\,\gamma \in \hat{\Gamma}_T, \text{ s.t. } b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1(\delta + \epsilon_0)}{4\sqrt{n}_x}, \forall k \geq 0\}.$$

We have $\mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{3,i} \cap \mathcal{E}_2)$ based on the same proof ideas of Lemma 14.

Next, we will show that

$$\Pr(\mathcal{E}_{3,k} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{T/m-1} G_{i,k}) \tag{11}$$

This is because for any $\gamma \in \hat{\Gamma}_T$, we have $b(\gamma z_t)^j \leq bw_t^j + \hat{w}_{\max}$ for any $b \in \{-1, 1\}$, $1 \leq j \leq n_x$, and $t \geq 0$. By $\mathcal{E}_{3,i}$, there exists $\gamma \in \hat{\Gamma}_T$ such that $b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1(\delta+\epsilon_0)}{4\sqrt{n}_x}$ for all $k \geq 0$. Thus, $b_{i,km+L_{i,k}}w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + \hat{w}_{\max} \geq \frac{a_1(\delta+\epsilon_0)}{4\sqrt{n}_x}$ for all k. Notice that this is equivalent with $b_{i,km+L_{i,k}}w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n}_x}$ for all k because $\epsilon_0 = \frac{4\sqrt{n}_x}{a_1}(\hat{w}_{\max} - w_{\max})$. In this way, we can prove (11).

Finally, we can complete the proof by the following.

$$\mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{3,i} \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k})$$

$$= \sum_{i=1}^{v_\gamma} \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0}) \cdots \mathbb{P}(G_{i,(T-1)/m-1} \mid \bigcap_{k=0}^{(T-1)/m-2} G_{i,k})$$

$$\leq \sum_{i=1}^{v_\gamma} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{(T-1)/m} \leq \text{Term 1}$$

where the second last inequality is by Lemma 18 and the last inequality uses the definition of v_γ in Lemma 7. □

G   Proof of Corollary 2

The proof of Corollary 2 is exactly the same as the proofs of Theorem 1 and Corollary 1. When A^* is stable, we can show that $\|x_t\|_2 \leq b_x$ for all t. Further, by Dean et al. [2019a], the sequence $\{x_t\}_{t\geq 0}$ satisfies the $(1, \sigma_x, p_x)$-BMSB condition. Therefore, we complete the proof.